Linear Optimization Under Uncertainty: Comparisons

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1. Introduction to Optimization Under Uncertainty

Part 1 of this presentation focuses on relationships among some fuzzy, possibilistic, stochastic, and deterministic optimization methods for solving linear programming problems. In particular, we look at several methods to solve one problem as a means of comparison and interpretation of the solutions among the methods.

OUTLINE: Part 1

1. Deterministic problem
2. Stochastic problem, stochastic optimization
3. Fuzzy problem – flexible constraints/goals, flexible programming
4. Fuzzy problem – fuzzy coefficients, possibilistic optimization
5. Fuzzy problem – Jamison&Lodwick approach

Definitions

Types of uncertainty

1. Deterministic – error which is a number
2. Interval – error which is an interval
3. Probabilistic – error which is a distribution, better yet are distribution bounds (see recent research of Lodwick&Jamison and Jamison&Lodwick
4. Possibilistic – error which is a possibility distribution, better are necessity/possibility bounds (see Jamison&Lodwick)
5. Fuzzy – errors which are membership function

Axioms

Confidence measures

$g(\emptyset)=0$

$g(\Omega)=1$

The weakest axiom that one could conceive to insure that the set function g has a minimum of conherence

$A \subseteq B \Rightarrow g(A) \leq g(B)$

When the reference set Ω is infinite, we impose continuity

For every nested sequence: $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ or

$A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$

$\lim_{n\to\infty} g(A_n) = g(_{n\to\infty} A_n)$

Measures of Possibility and of Necessity

Consequences of the axioms:

$$\forall A, B \subseteq \Omega, \qquad g(A \cup B) \geq \max(g(A), g(B))$$
$$g(A \cap B) \leq \min(g(A), g(B))$$

Thus we find, as the **limiting** case of confidence measure union is called (by Zadeh) *possibility* ***measure***

$$\forall A, B \subseteq \Omega, \ \Pi(A \cup B) = \max(\Pi(A), \Pi(B))$$

The **limiting** case of confidence measure intersection is called *necessity* ***measure***

$$\forall A, B \subseteq \Omega, \ \mathrm{N}(A \cap B) = \min(N(A), N(B))$$

Observations

- When A and B are disjoint

$A \cap B = \emptyset \Rightarrow \Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$

Probability for this case is:

$$p(A \cup B) = p(A) + p(B)$$

Thus, probability and possibility are different. In particular, suppose $A \cup B = \Omega$, $\Pi(\Omega) = \Pi(A \cup B)$

$$= \max(\Pi(A), \Pi(B))$$

$$= 1$$

$\Rightarrow \Pi(A) = 1$ or $\Pi(B) = 1$ or both.

- When E is a sure event such that:

$E \subseteq \Omega$, we can define a function Π for which

$\Pi(A) = 1$ if $A \cap E \neq \emptyset$

$\Pi(A) = 0$ otherwise

Then Π is a possibility measure.

Observations

- A function N can be constructed with values {0,1} from a sure event E, by:

$N(A)=1$ if $E \subseteq A$

$N(A)=0$ otherwise. Note that $N(A)=1$ means that A is sure (necessarily true).

The relation between necessity and possibility measures is:

$$\forall A \subseteq \Omega, \Pi(A) = 1 - N(\overline{A})$$

$$N(A) = 1 - \Pi(\overline{A})$$

$$\min(N(A), N(\overline{A})) = 0$$

$$\forall A \subseteq \Omega, \Pi(A) \geq N(A)$$

$$N(A) > 0 \Rightarrow \Pi(A) = 1$$

$$\Pi(A) < 1 \Rightarrow N(A) = 0$$

Possibility distribution

Possibility measures are ***set*** functions. We also need functions to act on individual elements ("points"). Thus,

$\pi(\omega)=\Pi(\{\omega\})$ when Π is defined.
$\Pi(A)=\sup\{\pi(\omega) \mid \omega\in A\}$ when π is defined.
π is a mapping of Ω into [0,1] called a possibility distribution.
It is normalized in the sense that
$\exists\ \omega,\ \pi(\omega)=1$ since $\Pi(\Omega)=1$.

Necessity distributions are defined in the same way.

The Deterministic Optimization Problem

The problem we consider is derived from the deterministic LP

$$\max \sum_{j=1}^{N} c_j x_j$$

$$\text{subject to: } \sum_{j=1}^{N} a_{ij} x_j \le b_i, \; 1 \le i \le M,$$

$x \ge 0$, where $a_{ij}, b_i,$ and c_j are real numbers.

Uncertainty and LP Models

Sources of uncertainty

1. **The inequalities –** flexible goals, vague goal, flexible programming, ***vagueness***
2. **The coefficients** – possibilistic optimization, ***ambiguity***
3. **Both in the inequalities and coefficients**

Optimization in a Fuzzy Environment – Bellman & Zadeh, "Decision making in a fuzzy environment," *Management Science*, 1970.

Let X be the set of alternatives that contain the solution of a given optimization problem; that is, the problem is feasible.

Let C_i be the fuzzy domain defined by the i^{th} constraint (i=1,…,m). For example, "United Airline pilots must have ***good vision***." In this case "good vision" is the associated fuzzy domain.

Let G_j be the fuzzy domain of the j^{th} goal (j=1,…,J). For example, "Profits must be ***high***." In this case "high" is the associated fuzzy domain.

Bellman & Zadeh called a fuzzy decision, the fuzzy set D on X

$$D = \left(\bigcap_{i=1}^{m} C_i\right) \cap \left(\bigcap_{j=1}^{J} G_j\right)$$

For example, Air Canada wants pilots with good vision and wants its profits high. Let *m* denote "membership function."

$$\forall x \in X, m_D(x) = \min\{m_{C_i}(x), m_{G_j}(x), i = 1 \ldots m, j = 1 \ldots J\}$$

The final decision, x_f, is chosen from the maximal decision set:

$$M_f = \{x_f \mid m_D(x_f) \geq m_D(x)\}$$

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When goals & constraints have ***unequal*** importance, membership functions can be weighted by x dependent coefficients as follows:

$\sum_{i=1}^{m}\alpha_i(x)+\sum_{j=1}^{J}\beta_j(x)=1$; that is, the weights are a convex combo

$$m_D(x)=\sum_{i=1}^{m}\alpha_i(x)m_{C_i}(x)+\sum_{j=1}^{J}\beta_j(x)m_{G_j}(x)$$

The fuzzy decision set *D* has the property that:

$$(\bigcap_{i=1}^{m}C_i)\bigcap(\bigcap_{j=1}^{J}G_j)\subseteq D\subseteq(\bigcup_{i=1}^{m}C_i)\bigcup(\bigcup_{j=1}^{J}G_i)$$

The definition of optimal decision as given by Zadeh & Bellman is not always satisfactory especially when $m_D(x_f)$ is very small (the graph is close to the *x-axis*). When this occurs goals and constraints are close to being contradictory (empty intersections). This issue is addressed in the sequel.

An Example Optimization Problem

We will use a simple example from Birge and Louveaux, page 4. A farmer has 500 acres on which to plant corn, sugar beets and wheat. The decision as to how many acres to plant of each crop must be made in the winter and implemented in the spring. Corn, sugar beets and wheat have an average yield of 3.0, 20 and 2.5 tons per acre respectively with a +/- 20% variation in the yields uniformly distributed. The planting costs of these crops are, respectively, 150, 230, and 260 dollars per acre and the selling prices are, respectively, 170, 150, and 36 dollars per ton. However, there is a less favorable selling price for sugar beets of 10 dollars per ton for any production over 6,000 tons. The farmer also has cattle that require a minimum of 240 and 200 tons of corn and wheat, respectively. The farmer can buy corn and wheat for 210 and 238 dollars per ton. The objective is to minimize costs. It is assumed that the costs and prices are crisp.

The Deterministic Model

$$\min 230x_1 + 260x_2 + 150x_3 - 150s_1 + 210p_1 -$$
$$36s1_2 - 10s2_2 - 170s_3 + 238p_3$$
$$\text{subject to: } x_1 + x_2 + x_3 = 500$$
$$y_{1j}x_1 - s_1 + p_1 \geq 240$$
$$y_{2j}x_2 - s1_2 - s2_2 \geq 0$$
$$s1_2 \leq 6000$$
$$y_{3j}x_3 - s_3 + p_3 \geq 200.$$

We take crop i = 1,2,3 to be corn, sugar beet, and wheat and yields y_{ij}, j=1,2,3 are low, average and high corresponding to crop i and all variables are non-negative.

The Stochastic Model

$\min c^t x + E_\xi \, Q(x,\xi)$

subject to: $Ax \leq b$,

$x \geq 0$,

$Q(x,\xi) = \min\{q^t y \,|\, Wy = h - Ty, y \geq 0\}$,

ξ is the vector with components of $q, h,$ and T,

E_ξ is the expectation with respect to ξ,

W is the fixed recourse matrix.

The Stochastic Model - Continued

For our problem we have:

$$Q(x,s) = \min\{q^t y = -150y_1 + 210y_2 - 36y_3 - 10y_4$$
$$-170y_5 + 238y_6\}$$
$$\text{subject to:}\; t_1(s)x_1 - y_1 + y_2 \geq 240$$
$$t_2(s)x_2 - y_3 - y_4 \geq 0$$
$$t_3(s)x_3 - y_5 + y_6 \geq 200.$$

Fuzzy LP – Tanaka (1974), Zimmermann (1976, 78)

Consider the standard LP:

$$\min z = c^T x$$

subject to: $Ax \leq b, \quad x \geq 0$

The standard "flexible" fuzzy LP is:

$$c^T x \underset{\sim}{<} z_0$$

$$A x \underset{\sim}{<} b, \quad x \geq 0$$

Let a_{ij}, b_i be "crisp" coefficients and define membership functions m_i , $i = 0, \ldots, M$ as follows:

$$m_i\left(\sum_{j=1}^{n} a_{ij} x_i\right) = \begin{cases} 1 \text{ for } \sum_{j=1}^{n} a_{ij} x_i \leq b_i \\ 1 - \frac{1}{d_i}\left(\sum_{j=1}^{n} a_{ij} x_i - b_i\right) \text{ for } b_i \leq \sum_{j=1}^{n} a_{ij} x_i \leq b_i + d_i \\ 0 \text{ for } \sum_{j=1}^{n} a_{ij} x_i > b_i + d_i \end{cases}$$

Fuzzy LP – Tanaka and Zimmermann's approach

where

$b_0 = z_0, \quad a_{0j} = c_j$

d_j are subjectively chosen (see radiation therapy problem)

These represent the amount of acceptable violation of each of the constraints. The initial constraint $b_0 = z_0$ is often determined by solving the standard LP and obtaining the optimal value and use this for z_0

< *figure*>

Fuzzy LP – Tanaka and Zimmermann’s approach

A fuzzy decision for the fuzzy LP is *D* such that:

$$m(x) = \min_i \{ m_i (\sum_j^n a_{ij} x_j) \}$$

The maximization of $m_D(x)$ is the equivalent “crisp” LP:

$\max\ x_{n+1}$ *** this is the maximization of m

$x_{n+1} \leq m_i(\sum_j^n a_{ij} x_j) \quad i = 0, \ldots, M$

$x_j \geq 0 \qquad j = 1, \ldots, n+1$

The constant $z_0 + d_0$ is determined by solving the above without constraint $i = 0$; let $\underline{x}$ be its solution, then $z_0 + d_0 = \sum_j^n c_j \underline{x}_j$ where z_0 is the optimal solution of the standard LP with b_i replaced by $b_i + d_i, i = 1, \ldots, M$.

Fuzzy LP - Tanaka, et.al., fuzzy in coefficients, possibilistic programming

max $c^t x$

subject to: $\sum_{j=1}^{N} [(1-\frac{1}{2}h)(a_{ij}+\bar{a}_{ij})+\frac{1}{2}h(a_{ij}-\underline{a}_{ij})]x_j \le$

$(1-\frac{1}{2}h)(b_i+\bar{b}_i)+\frac{1}{2}h(b_i-\underline{b}_i),\ 1\le i\le M$

$$\sum_{j=1}^{N} [\frac{1}{2}h(a_{ij}+\bar{a}_{ij})+(1-\frac{1}{2}h)(a_{ij}-\underline{a}_{ij})]x_j \le \frac{1}{2}h(b_i+\bar{b}_i)+(1-\frac{1}{2}h)(b_i-\underline{b}_i),\ 1\le i\le M$$

h is the level or degree of optimism for the satisfaction of the constraint.

Fuzzy LP - Tanaka, et.al. continued, possibilistic programming

Here a_{ij} and b_i are triangular fuzzy numbers

$\underline{a}_{ij} / a_{ij} / \overline{a}_{ij}$, $\underline{b}_i / b_i / \overline{b}_i$, c crisp, and $h \in [0,1]$.

Below h = 0.00, 0.25, 0.50, 0.75 and 1.00
is used.

Fuzzy LP – Inuiguchi, et. al., fuzzy coefficients, possibilistic programming

Necessity measure for constraint satisfaction

$$Nec\left(\sum_{j=1}^{N} \tilde{a}_{ij} x_j \le b_i\right) \ge h \leftrightarrow$$

$$\sum_{j=1}^{N} a_{ij} x_j + h \sum_{j=1}^{N} (\overline{a}_{ij} - a_{ij}) x_j \le b_i,$$

where $\tilde{a}_{ij} = \underline{a}_{ij} / a_{ij} / \overline{a}_{ij}$ and h$\in [0,1]$ is the degree to which the necessity of the constraint is satisfied.

Fuzzy LP – Inuiguchi, et. al. continued, possibilistic programming

Possibility measure for constraints

$$Pos\left(\sum_{j=1}^{N}\tilde{a}_{ij}x_j \le b_i\right) \ge h \leftrightarrow$$

$$\sum_{j=1}^{N}a_{ij}x_j - (1-h)\sum_{j=1}^{N}(\overline{a}_{ij} - a_{ij})x_j \le b_i,$$

where $h \in [0,1]$ is the degree to which the necessity of the constraint is satisfied.

Fuzzy LP – Jamison & Lodwick

Jamison&Lodwick consider the fuzzy LP constraints a penalty on the objective as follows:

$$\tilde{f}(x) = \tilde{c}^t x + \tilde{d}^t \max\{0, \tilde{b} - \tilde{A}x\},$$

$$\text{subject to: } Bx \leq d,$$

$$x \geq 0,$$

Fuzzy LP – Jamison & Lodwick, continued 2

The constraints are considered hard and the uncertainty is contained in the objective function. The expected average of this objective is minimized; that is,

$$\min\ F(x) = \frac{1}{2}\int_0^1 \{ f_\alpha^-(x) + f_\alpha^+(x)\}\, d\alpha$$

$$\text{subject to} \quad : Bx \le d,$$

$$x \ge 0.$$

Fuzzy LP – Jamison & Lodwick, continued 3

- F(x) is convex
- Maximization is not differentiable
- Integration over the maximization is differentiable
- We can make the integrand differentiable by transforming a max as follows:

$$\max\{0, x\} \to \frac{1}{2}(\sqrt{x^2 + \in} + x), \in > 0 \text{ small}$$

Table 1: Computational Results – Stochastic and Deterministic Cases

	Corn	Sugar Beets	Wheat	Profit ($)
LOW yield: **Acres planted – det.**	**25.0**	**375.0**	**100.0**	**$59,950**
AVERAGE yield: **Acres planted – det.**	**80.0**	**300.0**	**120.0**	**$118,600**
HIGH yield: **Acres planted – det.**	**66.7**	**250.0**	**183.3**	**$167,670**
Prob. of 1/3 for each yld. – discrete stochastic	**80.0**	**250.0**	**170.0**	**$108,390**
Recourse model – continuous stochastic	**85.1**	**279.1**	**135.8**	**$111,250**

Table 2: Computational Results – Tanaka, Ochihashi, and Asai

	Corn	Sugar Beets	Wheat	Profit ($)
Fuzzy LP **Acres planted, h = 0**	**72.8**	**272.7**	**154.5**	**$143,430**
Fuzzy LP **Acres planted, h = 0.25**	**71.8**	**269.2**	**159.0**	**$146,930**
Fuzzy LP **Acres planted, h = 0.50**	**70.6**	**264.7**	**164.7**	**$151,570**
Fuzzy LP **Acres planted, h = 0.75**	**69.0**	**258.6**	**172.4**	**$158,030**
Fuzzy LP **Acres planted, h = 1.00**	**66.7**	**250.0**	**183.3**	**$167,670**

Table 3: Computational Results – Necessity, Inuiguchi, et. al.

	Corn	Sugar Beets	Wheat	Profit ($)
Fuzzy LP, necessity > h **h = 0**	**80.0**	**300.0**	**120.0**	**$118,600**
Fuzzy LP, necessity > h **h = 0.25**	**76.2**	**285.7**	**138.1**	**$131,100**
Fuzzy LP, necessity > h **h = 0.50**	**72.8**	**272.7**	**154.6**	**$143,430**
Fuzzy LP, necessity > h **h = 0.75**	**69.6**	**260.9**	**169.5**	**$155,610**
Fuzzy LP, necessity > h **h = 1.00**	**66.7**	**250.0**	**183.3**	**$167,667**

Table 4: Computational Results – Possibility, Inuiguchi, et. al.

	Corn	Sugar Beets	Wheat	Profit ($)
Fuzzy LP, possibility > h h = 0.00	100.0	300.0	100.0	$100,000
Fuzzy LP, possibility > h h = 0.25	94.1	302.8	103.1	$103,380
Fuzzy LP, possibility > h h = 0.50	88.9	303.8	107.3	$107,520
Fuzzy LP, possibility > h h = 0.75	84.2	302.8	113.0	$112,550
Fuzzy LP, possibility > h h = 1.00	80.0	300.0	120.0	$118,600

Table 5: Computational Results – Jamison and Lodwick

	Corn	**Sugar Beets**	**Wheat**	**Profit ($)**
Fuzzy LP **Jamison & Lodwick**	**85.1**	**280.4**	**134.5**	**$111,240**
Recourse Model **Continuous Stochastic**	**85.1**	**279.1**	**135.8**	**$111,250**

Analysis of Numerical Results

- The extreme of the necessity measure, h=0, and the extreme of the possibility measure, h=1, generate the same solution which is the average yield scenario.
- Tanaka with h=0 (total lack of optimism) corresponds to the necessity h=0.5 model.
- Tanaka starts with a solution halfway between the deterministic average and high yield and ends up at the high yield solution.

Analysis of Numerical Results

- Possibility measure starts with a solution half way between the low and average yield deterministic and ends at the deterministic average yield solution.
- Necessity measure starts with the solution corresponding to average yield deterministic model and ends at the high yield solution.
- Lodwick & Jamison is most similar to the stochastic recourse optimization model yielding virtually identical solutions

- Complexity of the fuzzy LP using triangular or trapezoidal numbers corresponds to that of the deterministic LP.
- There is an overhead in the data structure conversion.
- The Lodwick&Jamison penalty approach is more complex than other fuzzy linear programming problems, especially since an integration rule must be used to evaluate the expected average.

- Complexity of Jamison & Lodwick corresponds to that of the recourse model with the addition of the evaluation of one integral per iteration.
- The penalty approach is simpler than stochastic programming in its modeling structure; that is, it can be modeled more simply. The transformation into a NLP using triangular or trapezoidal fuzzy numbers is straight forward.
- Used MATLAB and a 21-point Simpson’s integration rule.

2. Optimization Under Uncertainty -Methods and Applications in Radiation Therapy

The extension of flexible programming problems in order to allow for large “industrial strength” optimization is given.

Methods to handle large optimization under uncertainty problems and an application of these methods of to radiation therapy planning is presented. Two themes are developed in this study: (1) the modeling of inherent uncertainty of the problems and (2) the application of uncertainty optimization

Objectives of part 2 of this presentation

1. To demonstrate that fuzzy mathematical programming (fmp) is useful in solving large, “industrial-strength” problems
2. To demonstrate the usefulness and tractability of the Jamison & Lodwick and surprise approaches to fuzzy linear programming in solving large problems

OUTLINE – Part 2

I. Introduction: The radiation therapy treatment problem (RTP)

II. Modeling of uncertainty in the RTP

III. Optimization under uncertainty

  A. Zimmermann

  B. Inuiguchi, Tanaka, Ichihashi, Ramik, and others

  C. Jamison & Lodwick

  D. Surprise functions

IV. Numerical results – A, C and D

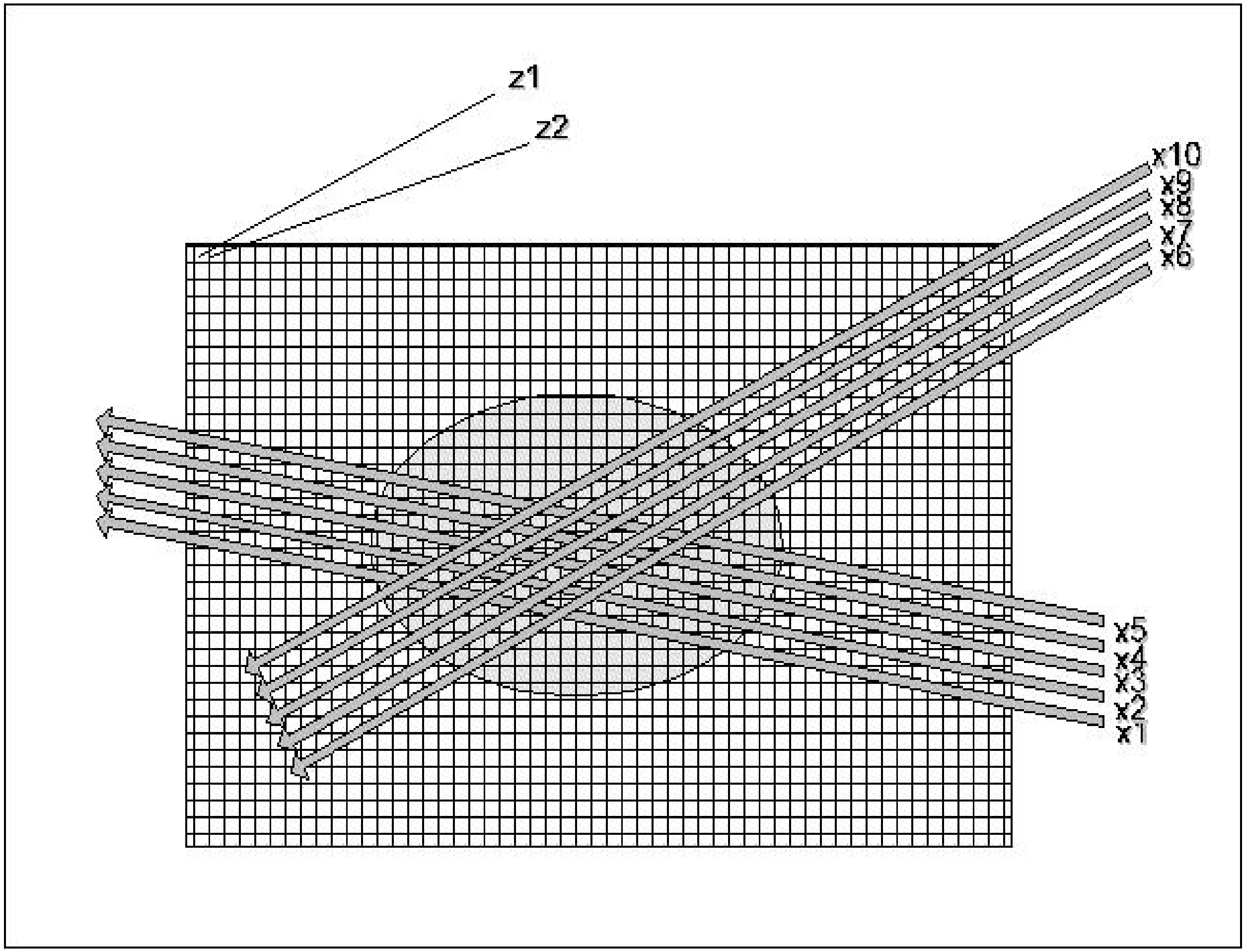
I. The Radiation Therapy Problem

- **The radiation therapy problem** (RTP) is to obtain, for a given radiation machine, a set of beam angles and beam intensities at these angles so that the delivered dosage destroys the tumor while sparing surrounding healthy tissue through which radiation must travel to intersect at the tumor.

I. Why Use a Fuzzy Approach?

- **Boundary between tumor and healthy tissue**
- **Minimum radiation value** for tumor a range of values
- **Maximum values** for healthy tissue a range of values
- The calculation of delivered dosage at a particular pixel is derived from a **mathematical model**
- **Alignment of the patient** at the time of radiation
- **Position of the tumor** at the time of radiation

I. CT Scan - Pixels and Pencils



I. ATTENUATION MATRIX

$$Ax = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \end{bmatrix}$$

$$= \begin{bmatrix} \text{total attenuated radiation at pixel 1} \\ \vdots \\ \text{total attenuated radiation at pixel m} \end{bmatrix} \leq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

I. EXAMPLE - Attenuation Matrix

- Suppose there are two pencils per beams and two voxels

$$Ax = \begin{bmatrix} 0 & 1 & 1 & 0.5 \\ 1 & 1 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 + x_3 + 0.5x_4 \\ x_1 + x_2 + 0.5x_3 \end{bmatrix} \le \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

I. Constraint Inequalities

$$\vec{\alpha}_{tumor}^{T} A\vec{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & .5 \\ 1 & 1 & .5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \left[\text{radiation at voxel 1}\right] = x_2 + x_3 + .5x_4 \geq z_1$$

I. Objective Functions

$$f(\vec{c},\vec{x}) = \vec{c}\cdot\vec{x} = \sum_{j}^{J} c_j x_j,$$

$$c_j \geq 0, \quad x_j \geq 0$$

Minimize total weighted radiation

I. The Fuzzy Optimization Model

$$\min f(\vec{x}) = \sum_{j}^{J} x_j \quad \text{(minimize total radiation)}$$

subject to:

$$\vec{\alpha}^{\mathrm{T}} A\vec{x} \geq \vec{T}_{\mathrm{min}} - \vec{p}_t,\ \vec{p}_t \geq 0,\ t \text{ indices of tumor voxels}$$

$$\vec{\alpha}^{\mathrm{T}} A\vec{x} \leq \vec{T}_{\mathrm{max}} + \vec{q}_t,\ \vec{q}_t \geq 0,$$

$$\vec{\alpha}_{c_k}^{T} A\vec{x} \leq \vec{d}_{\mathrm{max}} + \vec{r}_{c_k},\ \vec{r}_{c_k} \geq 0,\ \mathrm{c_k} \text{ indices of critical tissue}$$

$$\vec{\mathrm{x}} \geq 0$$

II. Modeling of uncertainty in the RTP

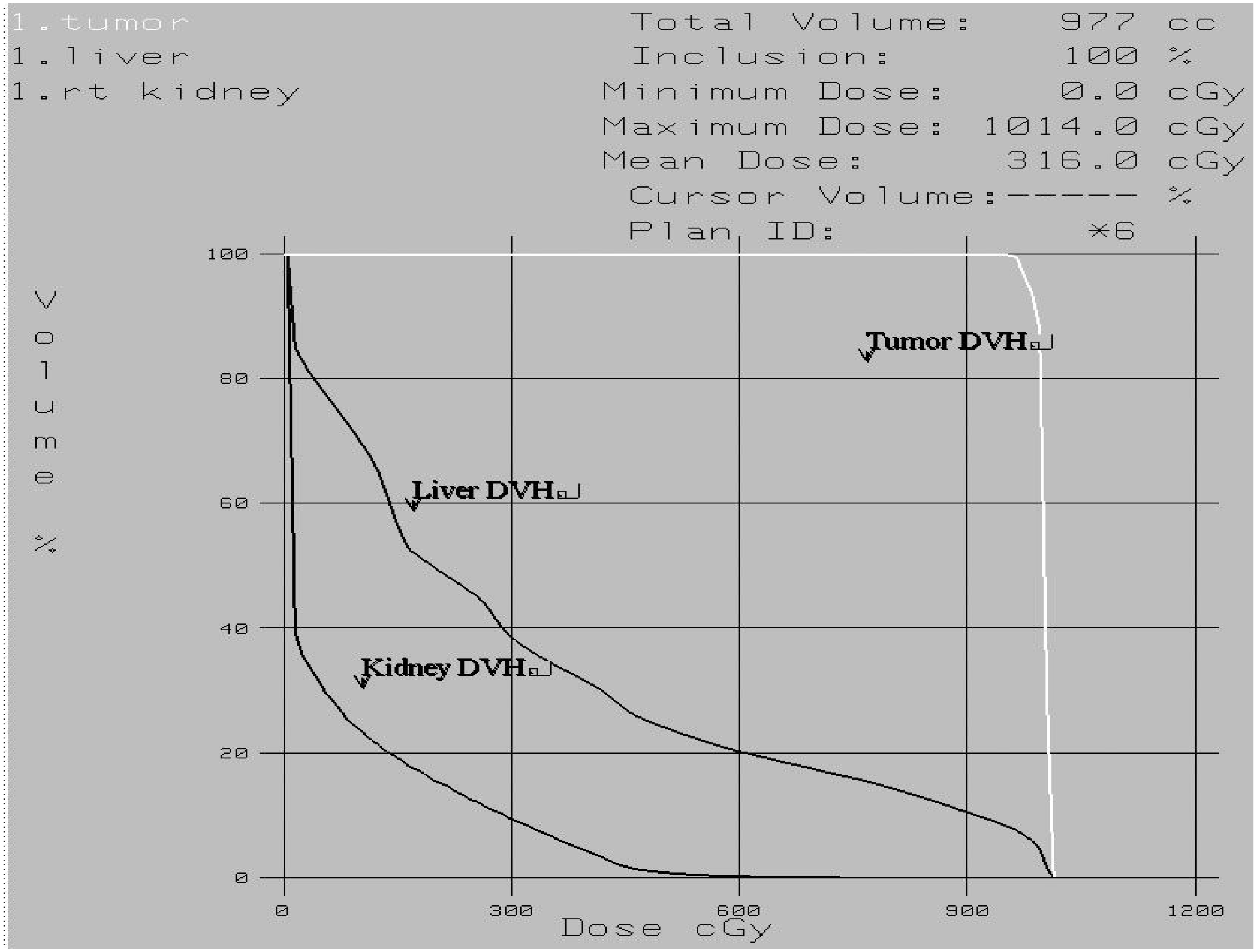
Four sources of uncertainty and fuzziness in the RTP:

1. Delineation of tumors and critical tissue
2. Radiation tolerances or critical dose levels for each tissue type or tumor
3. Model for the delivered dose, that is the dose transfer matrix
4. The location of tissue at the time of radiation

II. The RTP process – in practice

1. The oncologist delineates the tumor and critical structures
2. A candidate set of beam intensities is obtained; for example by linear programming, **fuzzy mathematical programming**, simulated annealing, or purely human choice.
3. These beam intensities are used as inputs to a FDA (Federal Drug Administration) approved dose calculator computer program to produce the graphical depiction of the dose deposition of each pixel (as color scales and dose-volume histograms, DVH’s – see Figure 1).

II. Example Dose Volume Histogram (DVH)



```
1.tumor                         Total Volume:     977 cc
1.liver                          Inclusion:        100 %
1.rt kidney                    Minimum Dose:       0.0 cGy
                               Maximum Dose:  1014.0 cGy
                               Mean Dose:        316.0 cGy
                                Cursor Volume:----- %
                                Plan ID:            *6
```

III. Optimization Under Uncertainty

The general fuzzy/possibilistic model considered here is:

$$\min c^T x$$
$$\text{subject to} : Bx \le \tilde{b}$$
$$0 \le x \le \bar{x} \text{ (max beam intensity)}$$

III. Zimmermann's approach

Translate to a real-value **linear program**

$$\max \ \lambda$$

subject to :

$$\lambda p_i + B_{i,\circ} x \le b_i + p_i \quad i = 1 \cdots m$$

$$0 \le x \le \bar{x}$$

Where $\tilde{b}$ is the fuzzy number $0/0/b/b + p$, trapezoid - membership

III. Jamison & Lodwick approach

Translate

$$\text{obj}: \tilde{F}(x) = f(x) + \tilde{p}^T \max\{0, Bx - \tilde{b}\}$$

$$subject\ to: 0 \le x \le \bar{x}$$

into the **nonlinear programming** problem

$$\max \quad \{ f(x) = EA(\tilde{F}) = \frac{1}{2}\int_0^1 (F_\alpha^-(x) + F_\alpha^+(x))\, d\alpha \}$$

$$\text{subject to} \quad : 0 \le x \le \bar{x}$$

III. Advantages to the J&L approach

1. If f(x) is convex, then the problem is a convex nlp with simple bound constraints
2. It optimizes over all alpha-levels; that is, it does not force each constraint to be at the same alpha-level
3. Large problems can be solved quickly; that is, it is tractable for large problems

III. Surprise function approach

Each fuzzy constraint

$$B_{i,\circ} x \leq \tilde{b}_i$$

is translated into a fuzzy equality constraint

$$B_{i,\circ} x = \tilde{\xi}_i$$

where the membership function $\mu_{\mathrm{i}}(\xi) = Pos(\tilde{b}_i \geq \xi)$.

Translate each membership function into a surprise function :

$$s_i(\xi) = ((\mu_{\mathrm{i}}(\xi))^{-1} - 1)^2$$

III. Surprise function approach - continued

The fuzzy problem is translated into the **nonlinear programming** problem

$$\min \quad \sum_{i} s_{i}(B_{i,\circ}\, x)$$

$$\text{subject to} \quad : 0 \le x \le \overline{x}$$

This is a convex nlp with simple bound constraints.

III. Why use the surprise function approach?

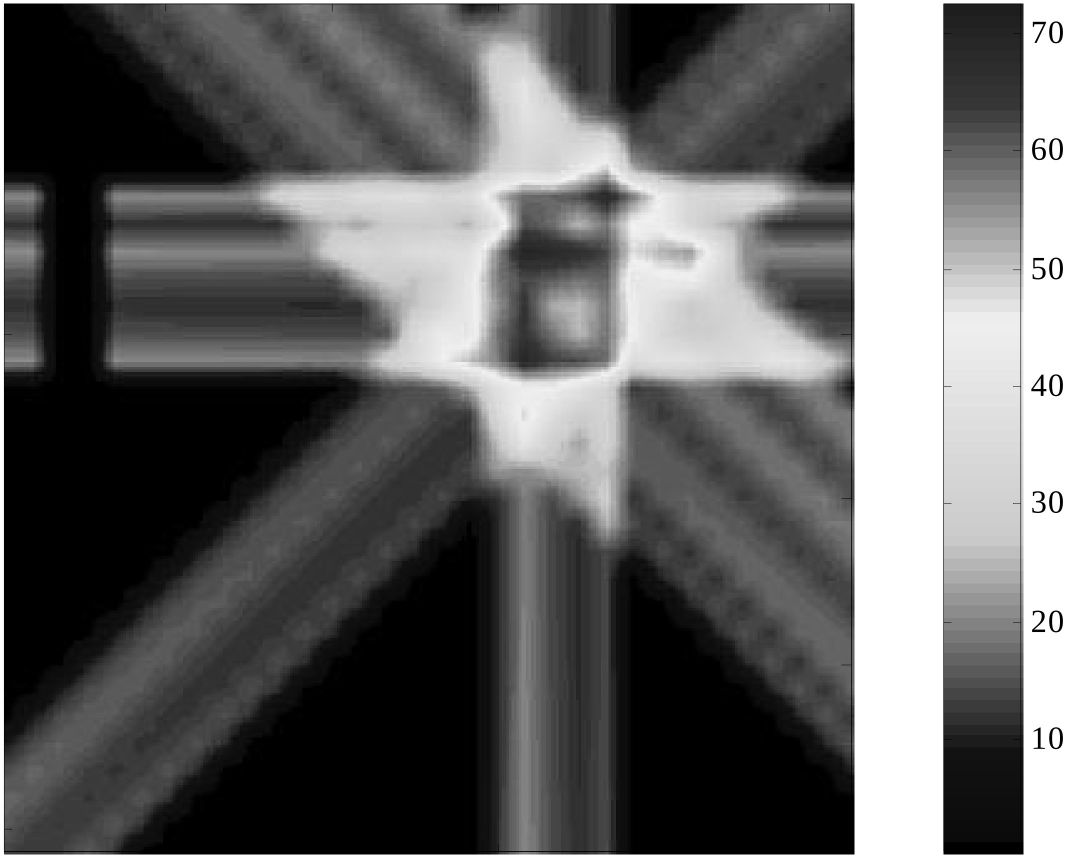
1. It is a convex nlp with simple bound constraints
2. It optimizes over all the alpha-levels; that is, it does not force each constraint to be a the same alpha-level
3. Large problems can be solved quickly; that is, it is tractable for large problems

IV. Surprise – problem: Black is out of body, blue is critical organ, yellow/green is other critical organs, red is tumor – 32x32 image, 8 angles

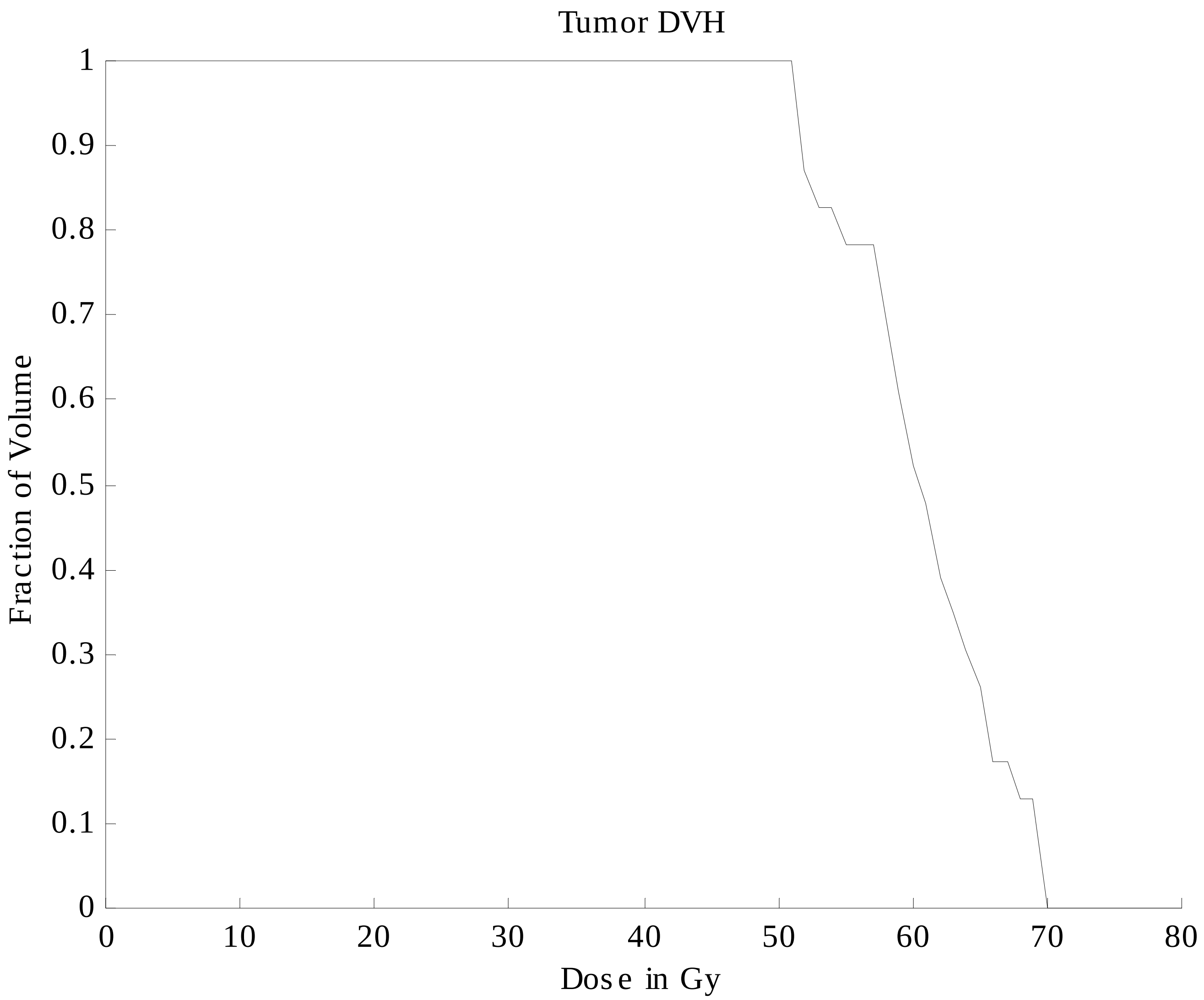
Set-up time =5.4580

Optimization time= 1.7130

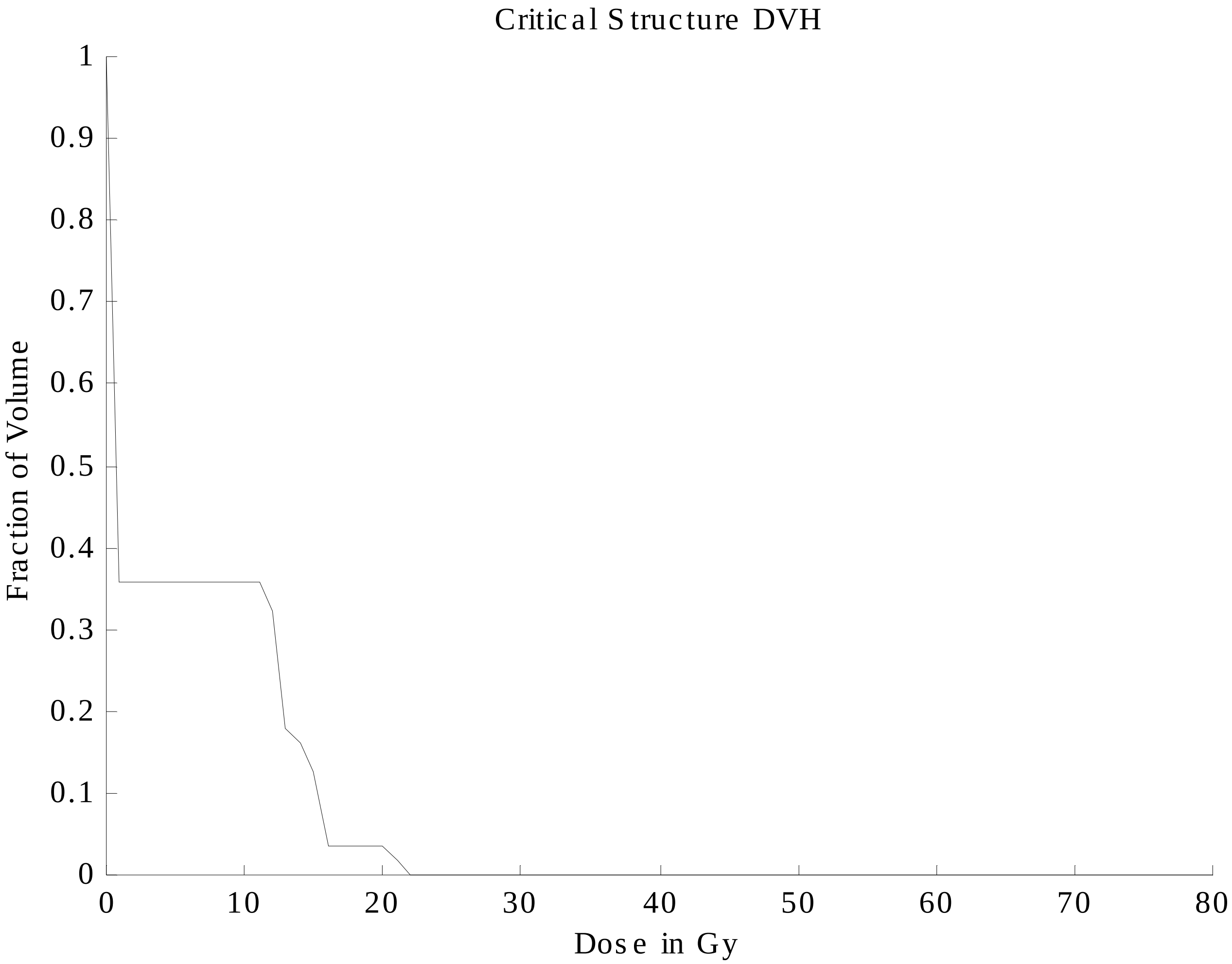
IV. Surprise 32x32 with 8 angles – delivered dosage



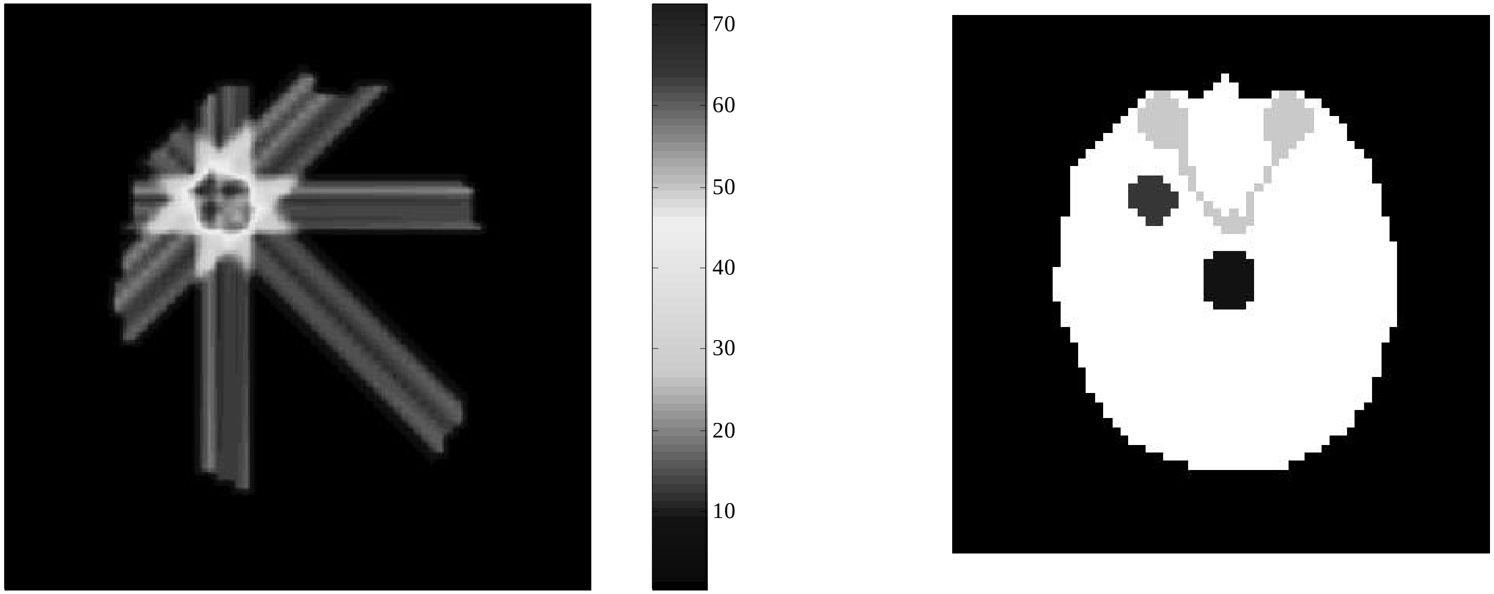
IV. Surprise 32x32 with 8 angles - Tumor dvh



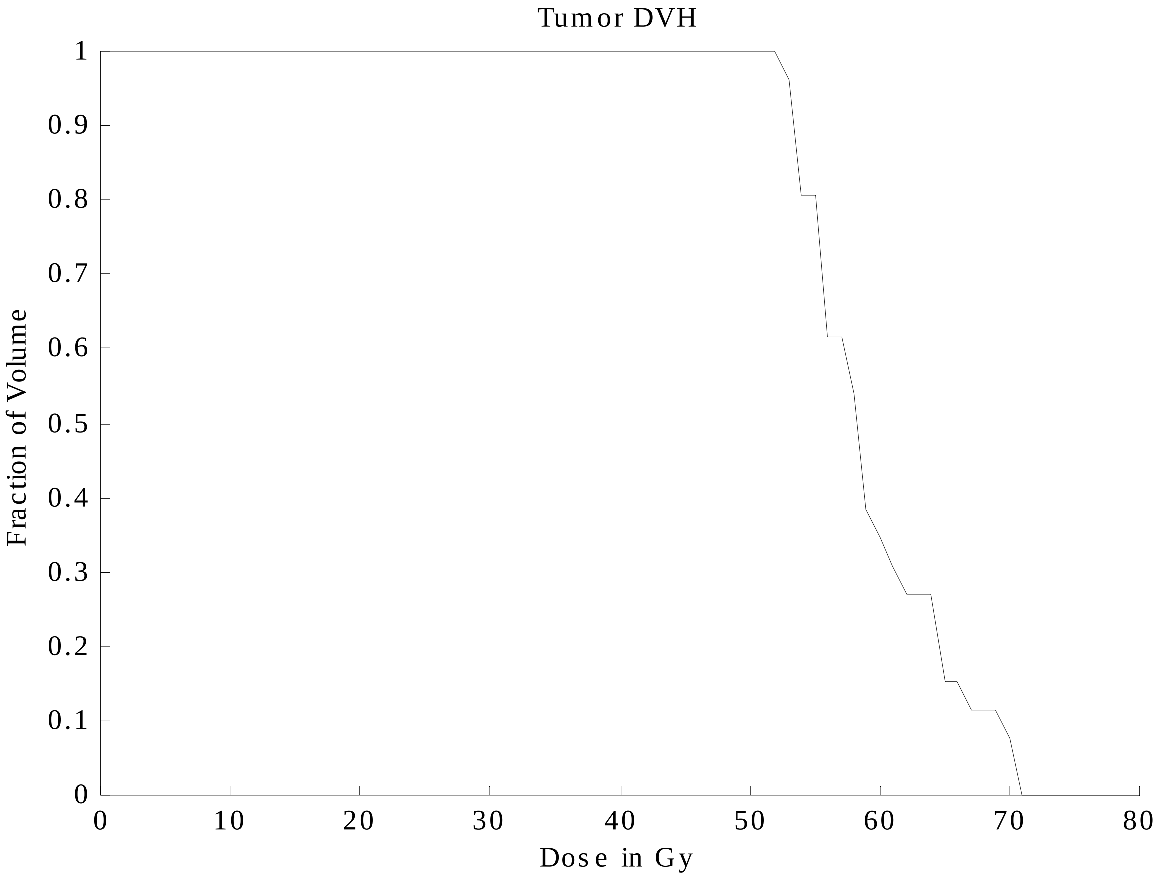
IV. Surprise 32x32 with 8 angles – Critical dvh



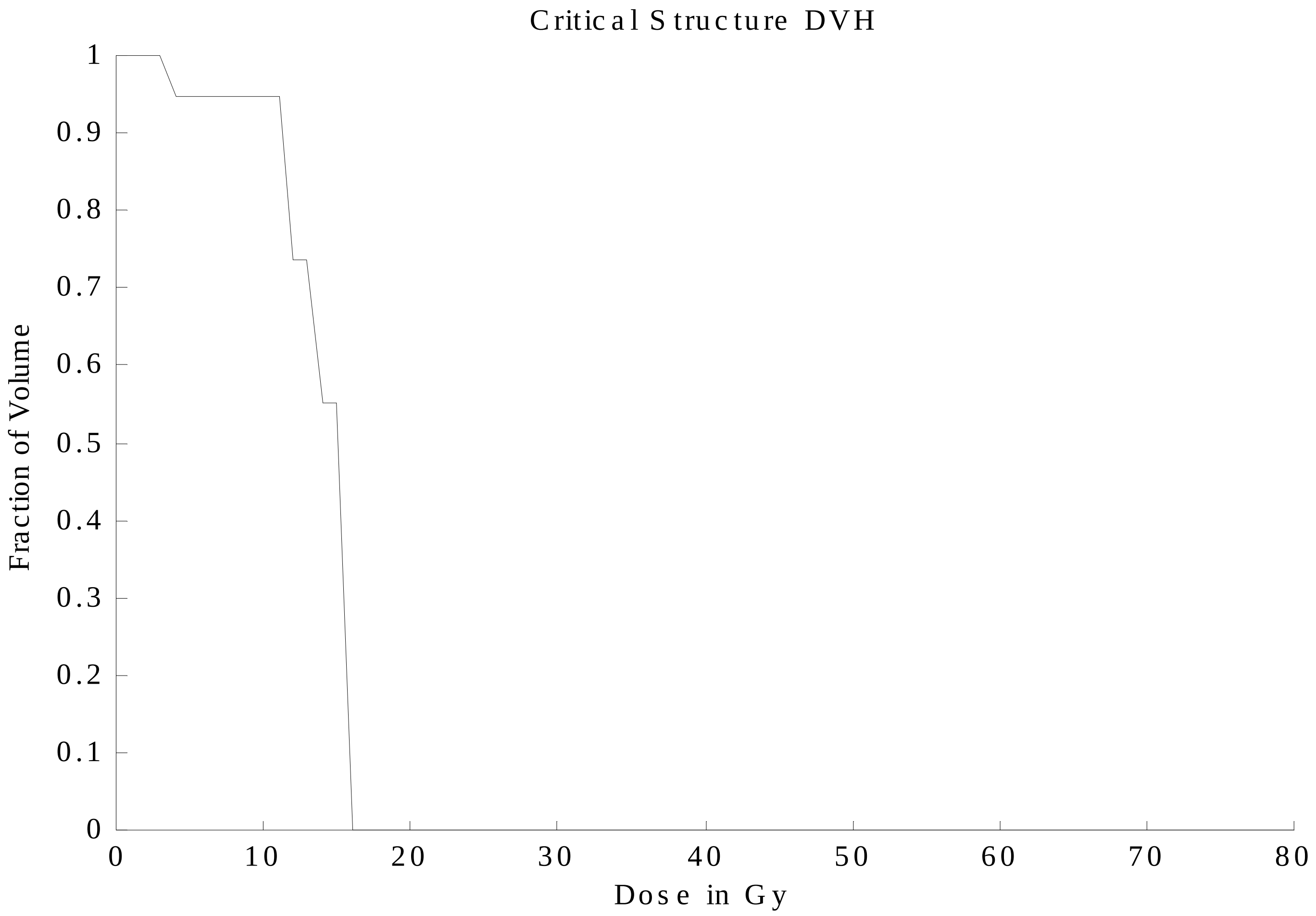
IV. Surprise 64x64 with 8 angles – delivered dosage
Set-up time = 11.0160, optimization time = 2.2930



IV. Surprise 64x64 with 8 angles – Tumor dvh



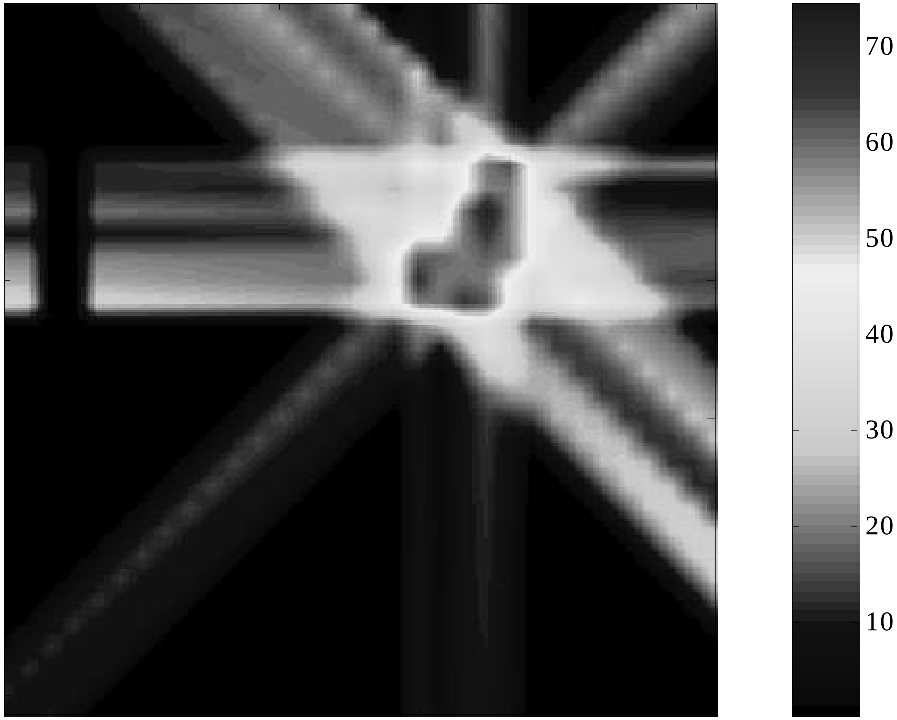
IV. Surprise 64x64 with 8 angles – Critical dvh



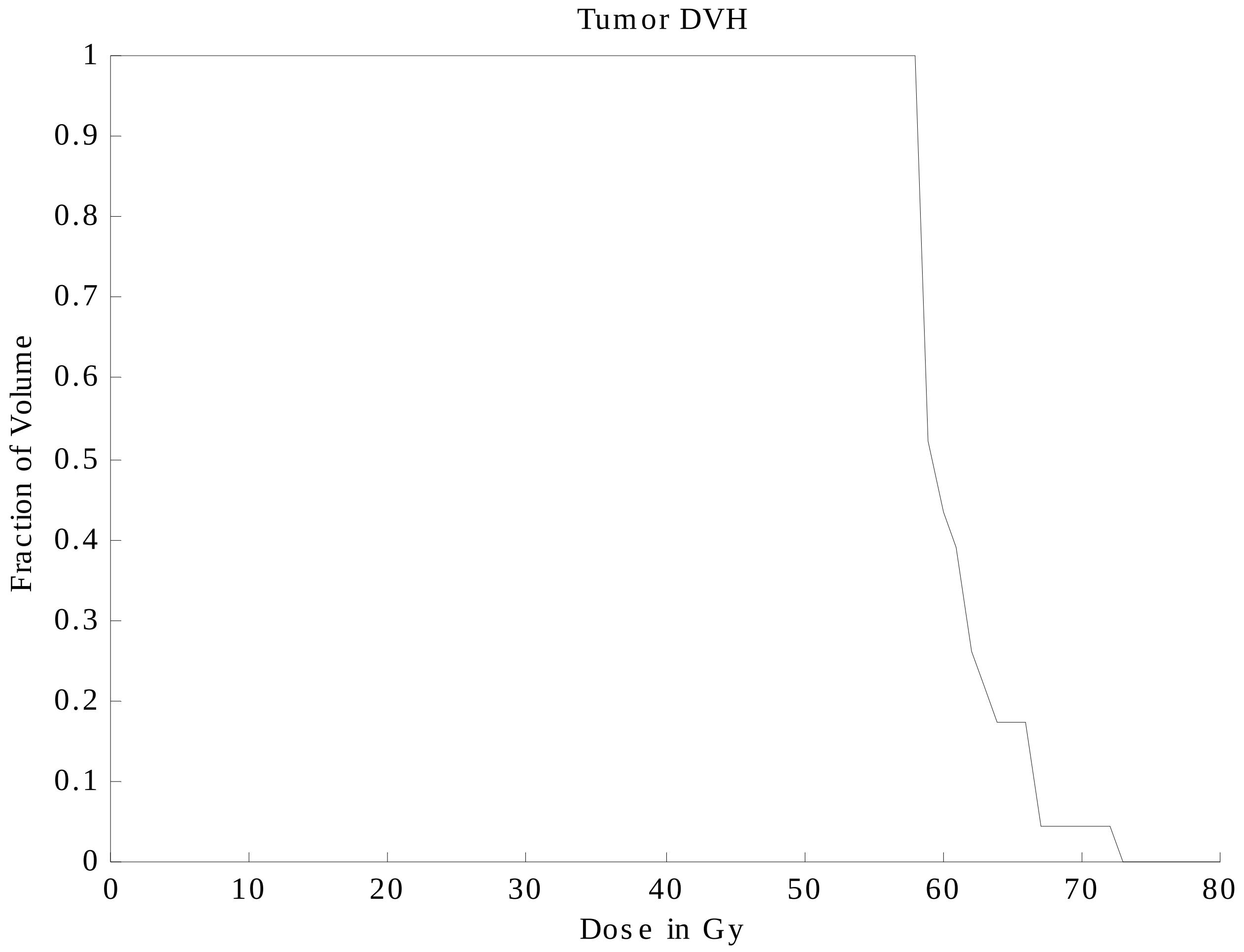
IV. Zimmermann – 32x32 with 8 angles

Set-up time = 4.6060

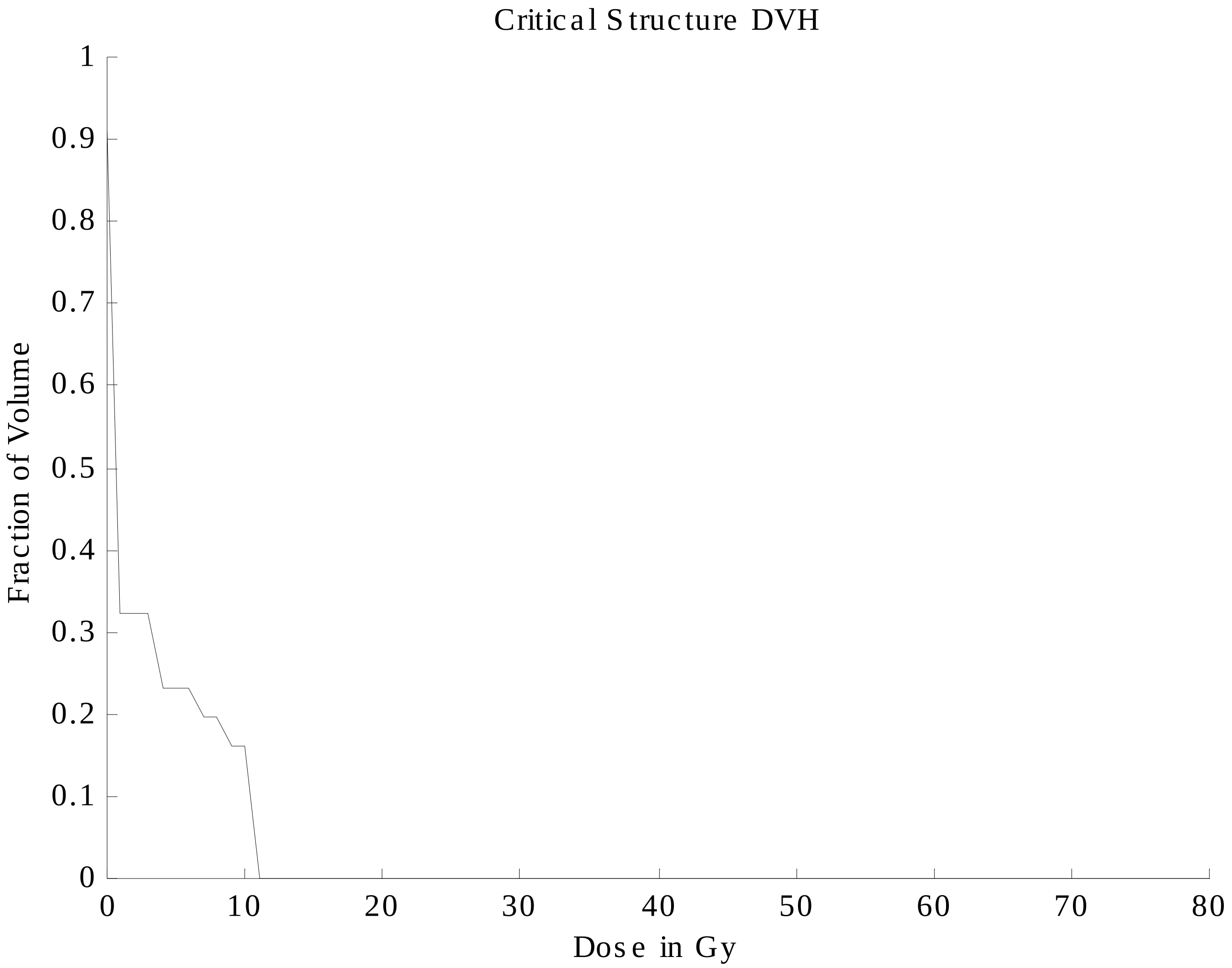
Opt time =171.1060



IV. Zimmermann 32x32 with 8 angles – tumor dvh

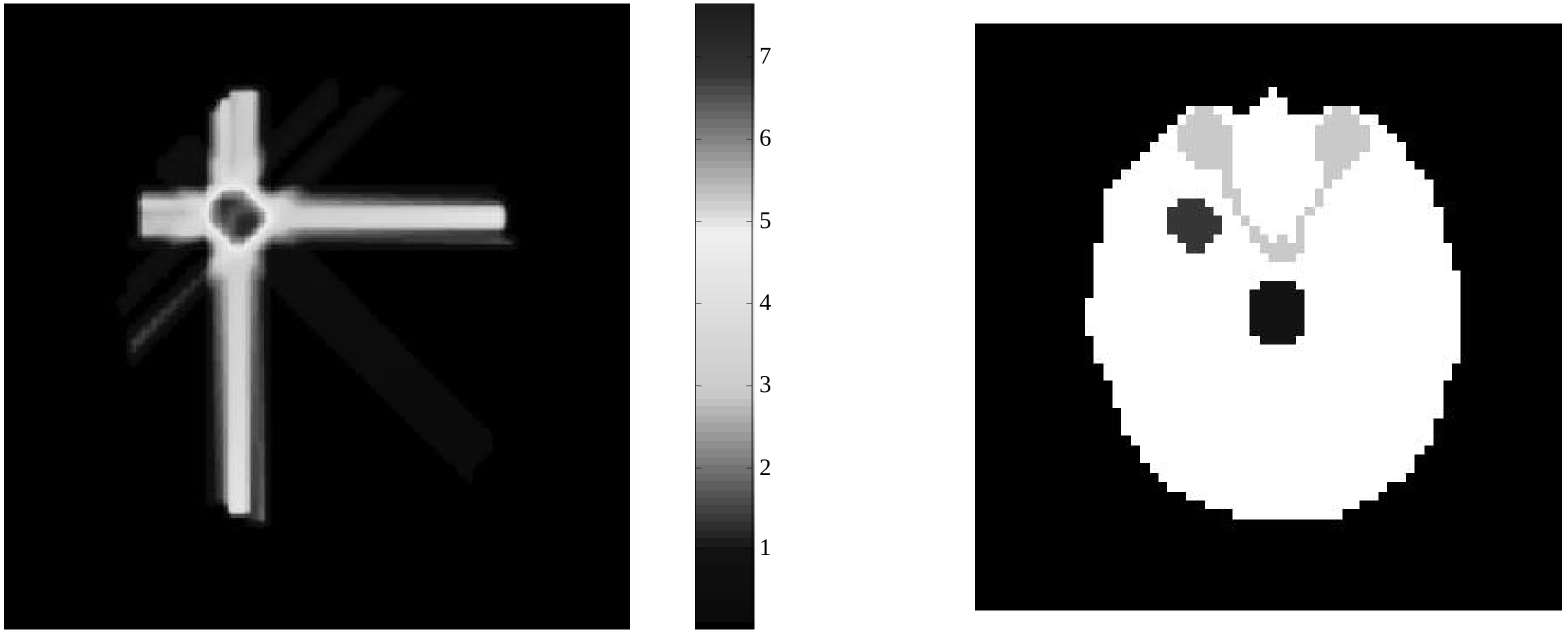


IV. Zimmermann 32x32 with 8 angles – critical dvh

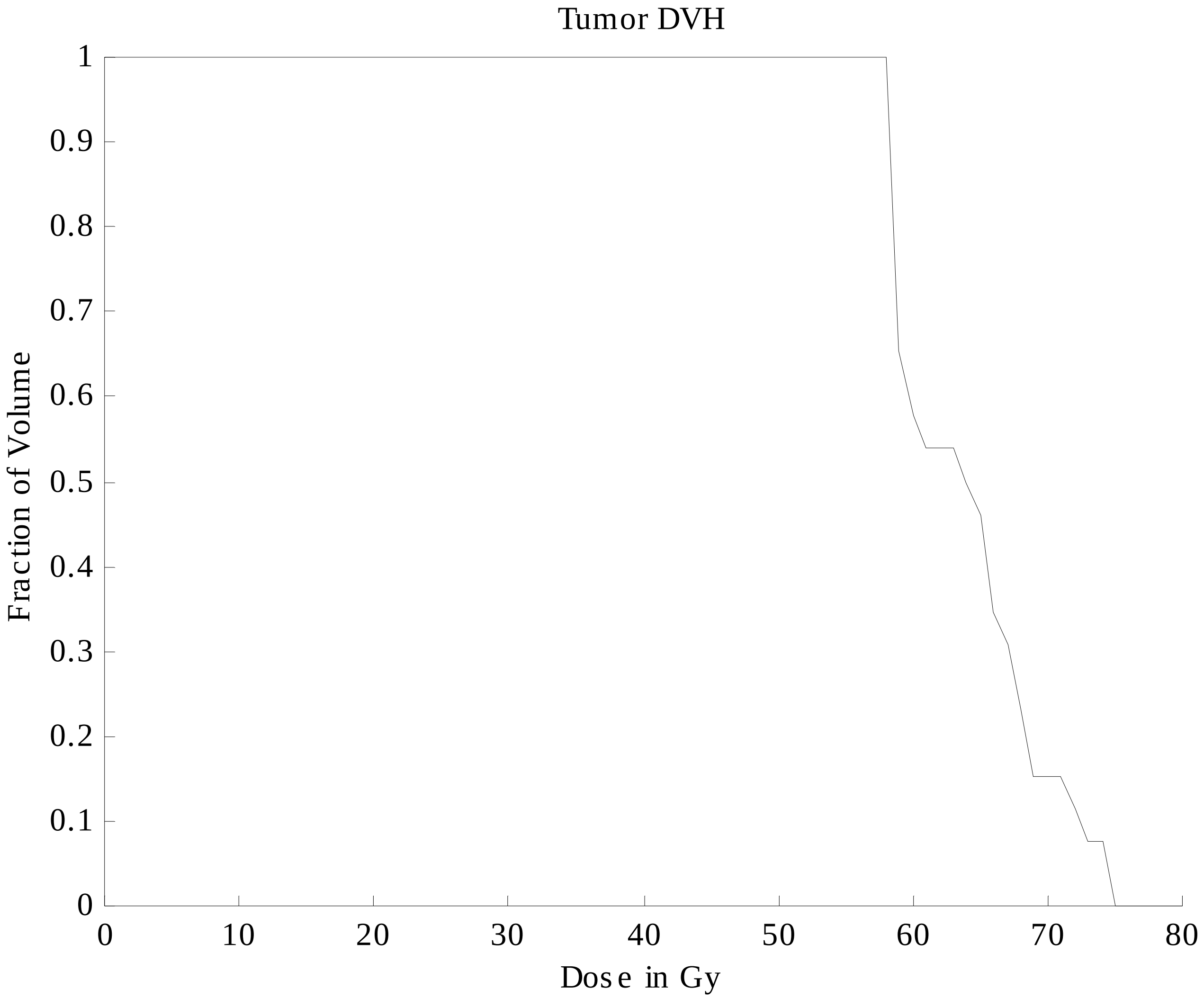


IV. Zimmermann – 64x64 with 8 angles

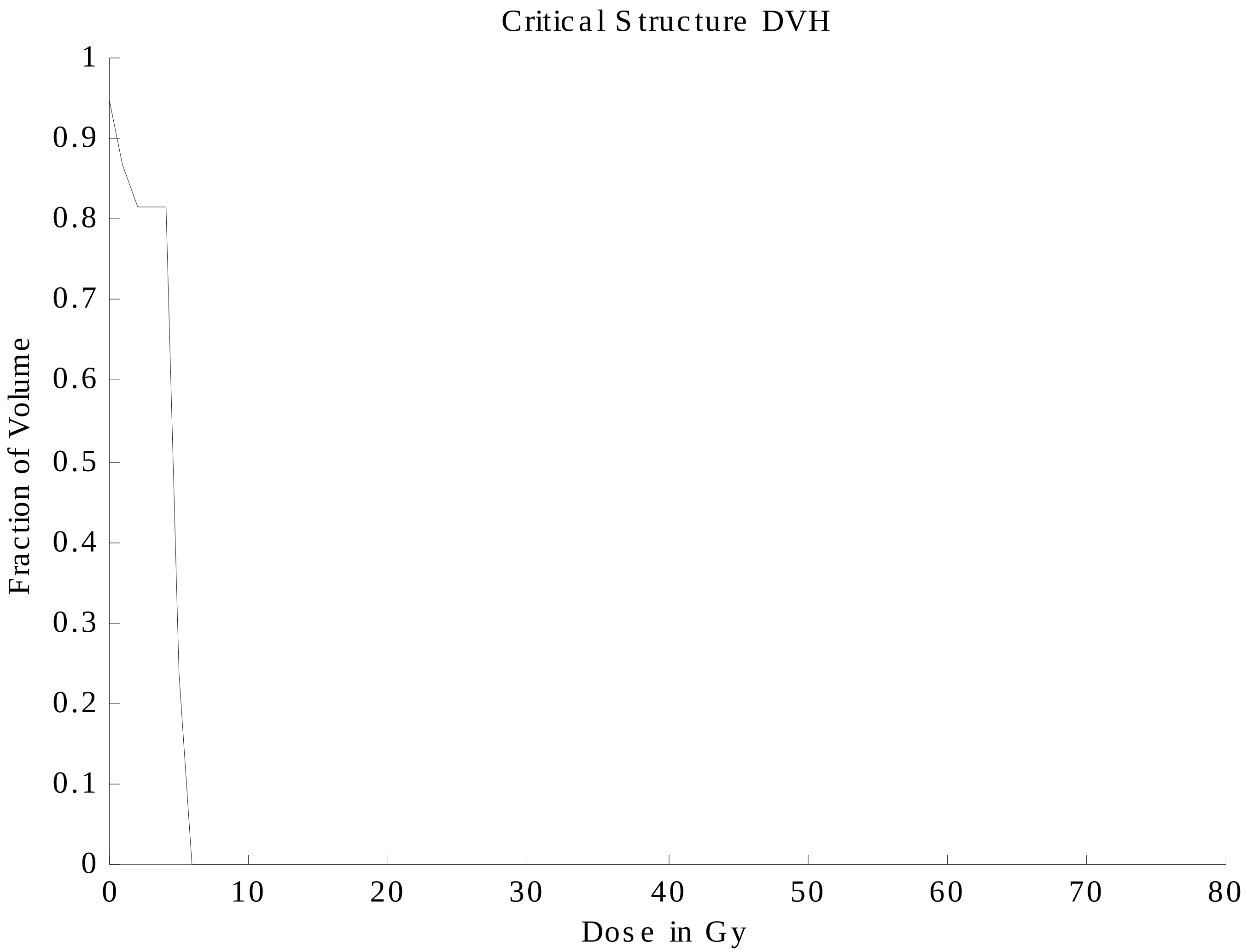
Set-up time = 8.8930, Optimization time =125.1100



IV. Zimmermann: 64x64 with 8 angles – Tumor dvh



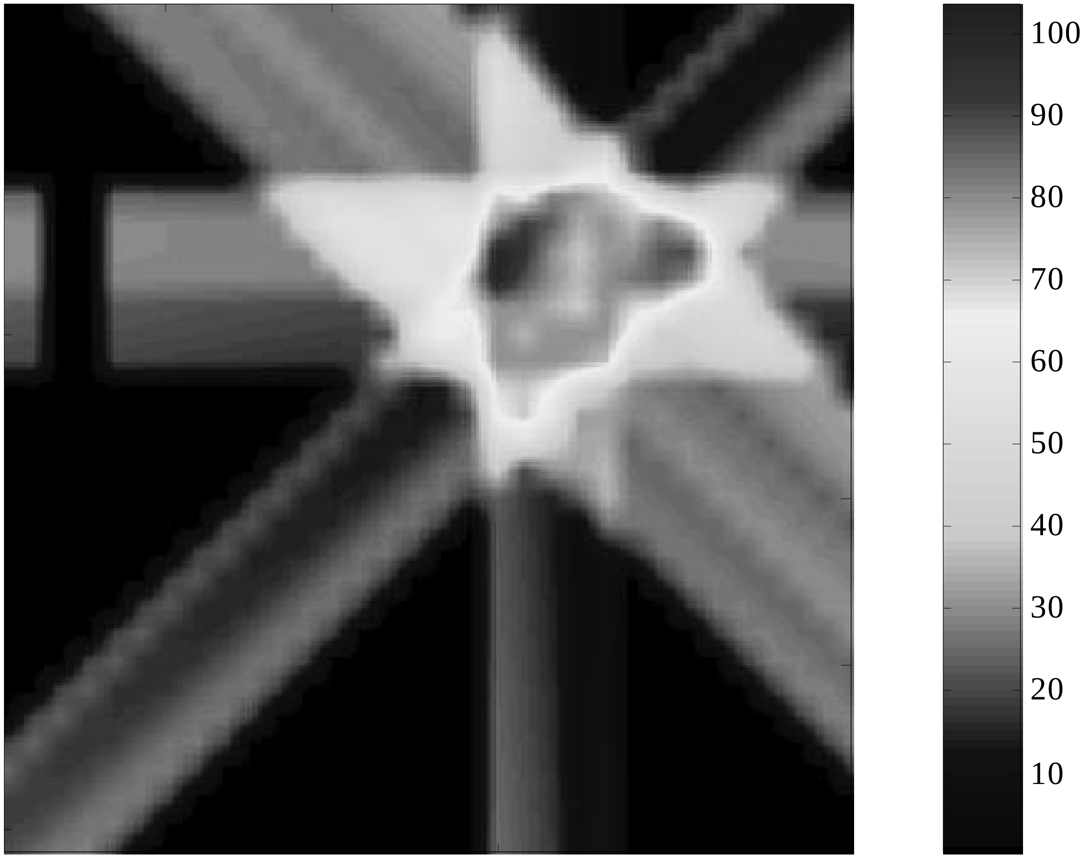
IV. Zimmermann: 64x64 with 8 angles – Critical dvh



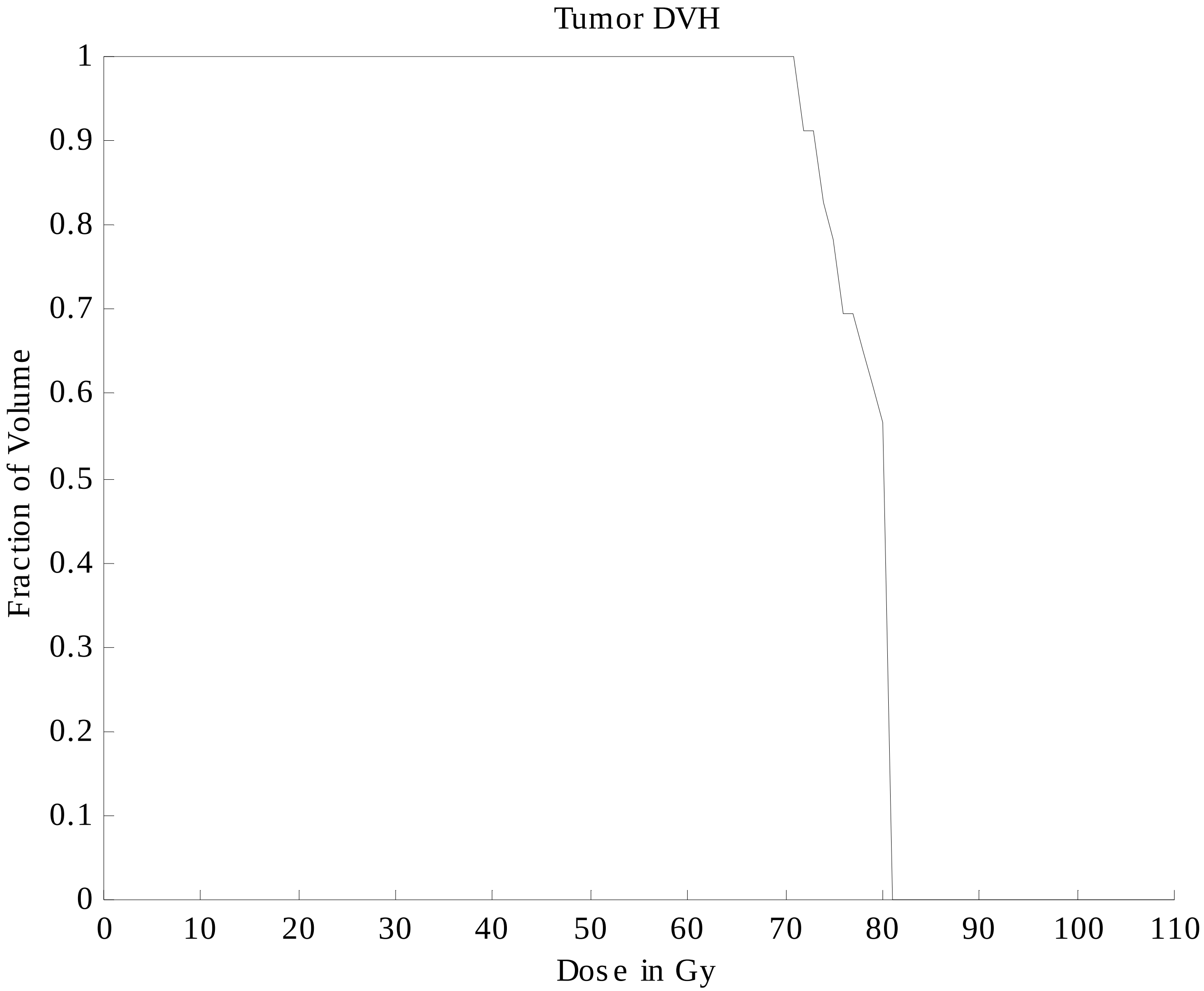
IV. J & L – 32x32 with 8 angles

Setup time - 5.3070

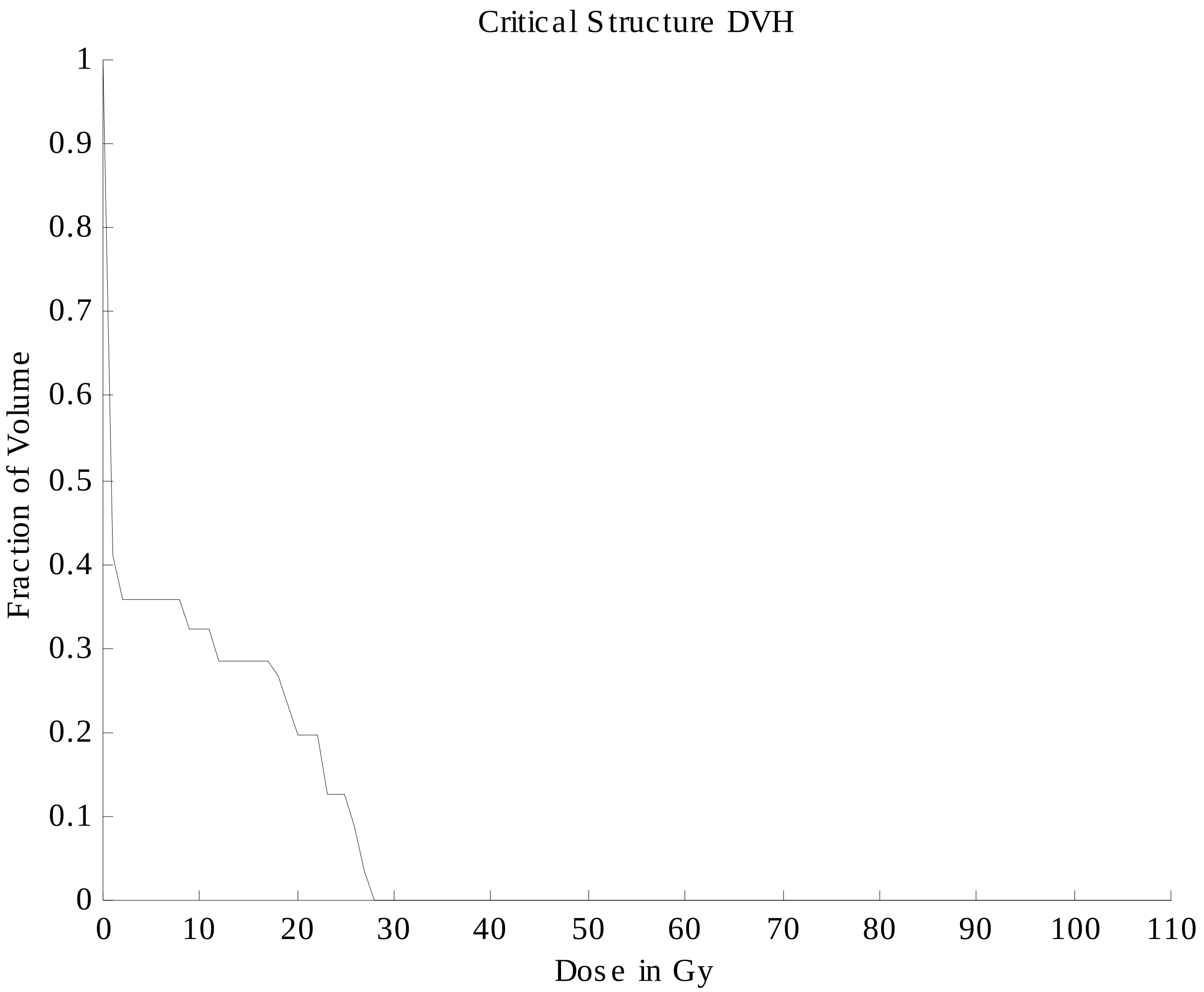
Optimization time - 7.3410



IV. J & L 32x32 with 8 angles – tumor dvh

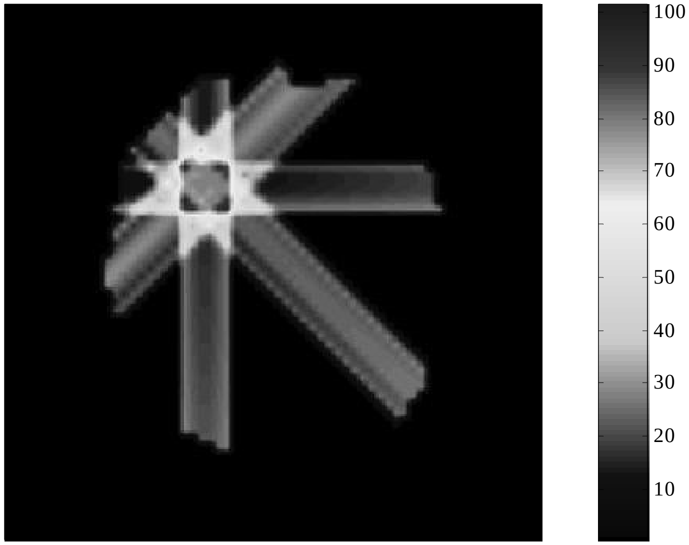


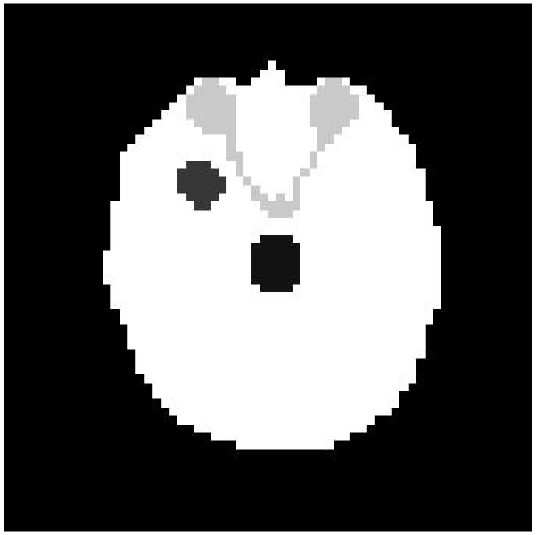
IV. J & L 32x32 with 8 angles – critical dvh



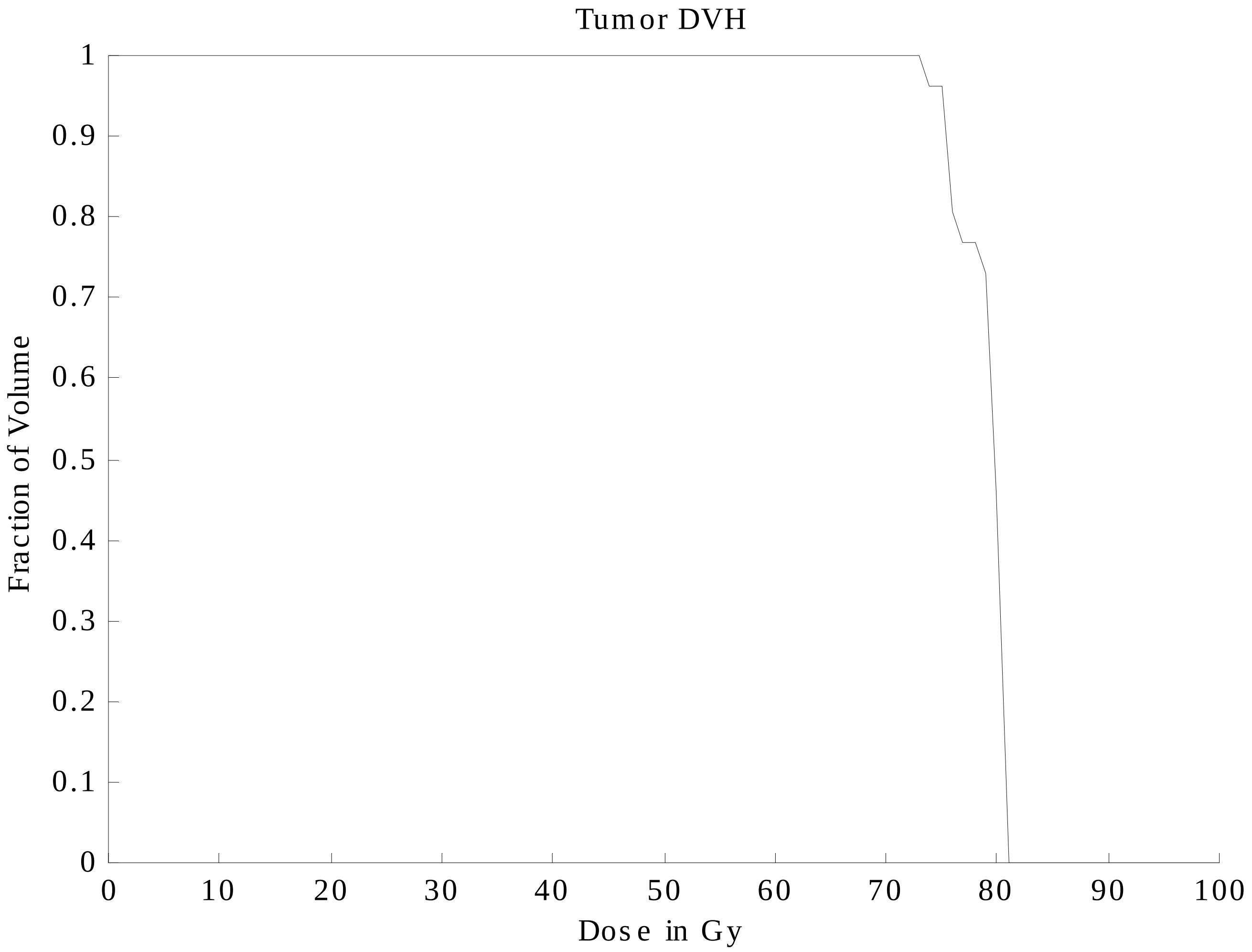
IV. J & L – 64x64 with 8 angles

Set-up time=13.0290, optimization time = 3.145





IV. J & L - 64x64 with 8 angles
Tumor dvh



IV. J & L - 64x64 with 8 angles
Critical structure dvh