Numerical Analysis I: MATH 4650/CS 4656 Midterm

Summer Semester: 1998 Name ______________________________

# Sample Test: Numerical Analysis I

INSTRUCTIONS - Show all work! Merely an answer may not give you credit. When the word ȷanalyze,ȷ ȷexplain,ȷ or ȷshowȷ are used, you are meant to explain why. In particular, you are to make relevant statements each of which is backed up by ȷlegalȷ facts. Please do not ask questions about this exam. If you think a problem is incorrect or insuŒciently clear, correct and/or clarify the problem explicitly writing down how you have corrected the problem and/or clariøed the problem and proceed with answering the question. If I agree with your correction and/or clariøcation, full credit will be given. Points for each problem part are listed next to the problems. Recall that to calculate with rounded 4-decimal digit arithmetic (abbreviated here as 4-digit arithmetic), for this class, means that numbers that are used are 4-digits long rounded to the nearest fourth signiøcant digit and rounded after each operation. Here is an example of 4-digit arithmetic:

$$\begin{aligned} 4.571 \times 3.99 + 0.8869 &= 18.24 + 0.8869 \\ &= 19.13 \end{aligned}$$

1. Let $y = 70 - \sqrt{4899}$ and $z = \frac{1}{70+\sqrt{4899}}$. Note that $y = z = 0.00714322161156$ and assume, for the purpose of calculating errors, that this is the exact value.

   (a) (5 points) Using 4-digit arithmetic (rounding) compute the value of $y$.

   (b) (5 points) Calculate the absolute error in your 4-digit result for $y$.

   (c) (5 points) Using 4-digit arithmetic (rounding) compute the value of $z$.

   (d) (5 points) Calculate the absolute error in your 4-digit result for $z$.

   (e) (5 points) Explain why the absolute errors in the 4-digit calculations for $y$ and $z$ are diœerent. Note: Since the roundoœ in both calculations are the same (they are both done with 4-digit arithmetic), roundoœ is not the reason especially since computing $z$ requires one more operation which would say that $z$ should have more error from a pure roundoœ error argument. Be speciøc and point out exactly where in the calculation of $y$ the problem occurs.

2. (25 points) The following table of numbers is generated from three methods (bisection method, Newton's method and secant method) for solving $f(x) = 0$, where $f(3) = 0$.

| | *Method* 1 | *Method* 2 | *Method* 3 |
|---|---|---|---|
| *Iteration* 1 | 4 | 3.75 | 3.65 |
| *Iteration* 2 | 3.690065559342354 | 3.61875 | 3.54787330161046 |
| *Iteration* 3 | 3.476190476190476 | 3.42113671875 | 3.415497356923732 |
| *Iteration* 4 | 3.328602647305882 | 3.195091749467468 | 3.265597005554780 |
| *Iteration* 5 | 3.226757369614512 | 3.041866869781306 | 3.128755624981744 |
| *Iteration* 6 | 3.15647745109804 | 3.001928118263814 | 3.039899197162600 |
| *Iteration* 7 | 3.107979699816434 | 3.000004089404044 | 3.005993839918846 |
| *Iteration* 8 | 3.074513071951448 | 3.000000000018396 | 3.000279025611330 |

Determine whether method 1, 2, and 3 is the bisection, Newton or secant method solely based on respective rates of convergence. You cannot assume that if you are able to determine two of the methods that the third is the missing method since two or all three of the methods may be the same.

(a) Method 1 is ………… because

Method 2 is ………… because

Method 3 is ………… because

3. Consider solving the following equation (the associated graph is given below).

$$\begin{aligned} f(x) &= x^4 - 7x^3 + 15x^2 - 13x + 4 = 0 \\ x &\in [0, 5] \end{aligned}$$

Note that, $f(x) = (x - 4.0)(x - 1.0)^3$.

(a) (5 points) To ønd the root at $x = 1$, pick a correct starting interval $[a, b]$ for the bisection method. Explain why the $a$ and $b$ you have chosen will converge to the solution $x = 1$.

(b) (5 points) Suppose the starting interval $[a, b]$ for the bisection method to ønd the root at $x = 4$ is $[3.5, 4.25]$, how many iterations of the bisection method are required to guarantee an accuracy of $10^{-8}$?

(c) (5 points) In applying Newton's method to ønd $f(x) = 0$, suppose the starting point is $x_0 = 3.24$, to what root (if any) will the Newton's method converge? Explain.

(d) (5 points) In applying Newton's method to ønd $f(x) = 0$, suppose the starting point is $x_0 = 3.25$, to what root (if any) will the Newton's method converge? Explain.

(e) (5 points) In applying Newton's method to ønd $f(x) = 0$, suppose the starting point is $x_0 = 3.26$, to what root (if any) will the Newton's method converge? Explain.

4. Suppose we have the following data set:

$$D_3 = \{(0, 1), (1, 4), (2, 8)\}.$$

(a) (5 points) Verify that the following are the correct $L_{2,j}(x)$ for the $D_3$ above. You don't have to compute these but just verify the appropriate associated properties.

$$\begin{aligned} L_{2,0}(x) &= \tfrac{1}{2}x^2 - \tfrac{3}{2}x + 1 \\ L_{2,1}(x) &= -x^2 + 2x \\ L_{3,1}(x) &= \tfrac{1}{2}x^2 - \tfrac{1}{2}x \end{aligned}$$

(b) (5 points) Compute the Lagrange interpolating polynomial, $p_2(x)$, associated with $D_3$.

(c) (5 points) The table associated with the Newton form of the interpolating polynomial associated with $D_3$ is:

$$\begin{array}{cccc} x & y & & \\ 0 & 1 & & \\ & & \frac{4-1}{1-0} = 3 & \\ 1 & 4 & & \frac{4-3}{2-0} = \frac{1}{2} \\ & & \frac{8-4}{2-1} = 4 & \\ 2 & 8 & & \end{array}$$

so that the degree two interpolating polynomial $q_2(x)$ is:

$$q_2(x) = 1 + 3(x-0) + \frac{1}{2}(x-0)(x-1).$$

Show that $q_2(x)$ above is in fact the interpolating polynomial for $D_3$.

(d) (5 points) Suppose now that we add one point to $D_3$ to obtain $D_4 = \{(0,1),(1,4),(2,8),(3,9)\}$ and we have the Newton form table:

$$\begin{array}{ccccc} x & y & & & \\ 0 & 1 & & & \\ & & \frac{4-1}{1-0} = 3 & & \\ 1 & 4 & & \frac{4-3}{2-0} = \frac{1}{2} & \\ & & \frac{8-4}{2-1} = 4 & & \cdots\cdots \\ & & & \cdots\cdots & \\ 2 & 8 & & & \\ & & \cdots\cdots & & \\ 3 & 9 & & & \end{array}$$

Complete this table.

5. Let

$$f(x) = 3x^2 - 4x.$$

(a) (10 points) Below is MATLAB's computation of the numerical derivative using $(f(x+h) - f(x))/h$ at $x = 1$ for various values of h. Recall that MATLAB has 17 digit precision.

| h | $(f(1+h) - f(1))/h$ |
|---|---|
| 1e-001 | 2.300000000000008 |

| | |
|---|---|
| 1e-002 | 2.029999999999976 |
| 1e-003 | 2.002999999998867 |
| 1e-004 | 2.000299999997956 |
| 1e-005 | 2.000030000059994 |
| 1e-006 | 2.000002999658079 |
| 1e-007 | 2.000000303148396 |
| 1e-008 | 1.999999943436137 |
| 1e-009 | 2.000000609569952 |
| 1e-010 | 2.000000165480742 |
| 1e-011 | 2.000000165480742 |
| 1e-012 | 2.000621890374532 |
| 1e-013 | 1.993960552226781 |
| 1e-014 | 1.998401444325282 |
| 1e-015 | 2.664535259100376 |
| 1e-016 | -4.440892098500626 |
| 1e-017 | 0 |

Explain why the derivative is improves until h $= 10^{-8}$ and then gets worse. Why is the derivative negative (the sign is wrong) for h=$10^{-16}$? Why is the derivative zero for h=$10^{-17}$?

(b) (10 points) Compute the derivative of $f(x)$ at $x = 1$ using forward automatic diœerentiation.

(c) (10 points) Compute the derivative of $f(x)$ at $x = 1$ using backward automatic diœerentiation.