Sample final exam-Math 4650/CS 4656: Numerical Analysis I - Final Exam

**INSTRUCTIONS - Please show all work** since you will lose at least half the credit if you do not show your work. When the words "analyze," "explain," or "show" are used, you are meant to explain why. In particular, you are to make relevant statements each of which is backed up by "legal" facts. Recall that to calculate with rounded 3-decimal digit arithmetic (abbreviated here as 3-digit arithmetic), for this class, means that numbers have 3 significant digits rounded to the nearest third significant digit and rounded after each operation. Here is an example of 3-digit arithmetic:

$$\begin{aligned} 4.61 \times 3.99 + 0.886 &= 18.4 + 0.886 \\ &= 19.3 \end{aligned}$$

**Note:** There are 140 points possible; that is, you have 15 extra points.

1. Given the data set

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_i$ | 0 | .25 | .5 | .75 | 1 |
| $x_i$ | 2 | 1 | 0 | $-1$ | 1 |
| $y_i$ | 0 | 1 | 2 | 0 | $-1$ |

   (a) (5 points) Draw a smooth curve going through these points as if these were points that a mouse went through in a paintbrush drawing.

   (b) (5 points) Explain why parametric curve methods (from section 3.5 for example) **must** be used to render these points (as opposed to Lagrange interpolating polynomials or cubic splines for example).

2. Suppose that the Hermite cubic for the x-coordinate points of a Bezier curve is

$$p(t) = x(t) = 4t^3 - 3t^2 + 2t + 1$$

   (a) (5 points) Complete the following forward difference table (associated with $p(t)$ given above)

| $t$ | $p(t)$ | $r_2(t)$ | $r_1(t)$ | $r_0(t)$ |
|---|---|---|---|---|
| 0 | 1 | 0.174 | $-0.036$ | 0.024 |
| .1 | | | | |
| .2 | | | | |

   (b) (5 points) **How** is forward differencing used in conjuction with Bezier curves? Explain by making reference to $p(t)$ as given for this problem. Recall how forward differencing was used in project 3.

   (c) (5 points) **Why** would forward differencing be used in conjunction with Bezier curves (as opposed to evaluating with $p(t)$ directly)? Explain by making reference to $p(t)$ as given for this problem.

3. The data set for $f(x) = \sin x$ on the interval $[0, \pi]$ is:

| $x_i$ | 0 | $\pi/3$ | $2\pi/3$ | $\pi$ |
|---|---|---|---|---|
| $y_i$ | 0 | $\sqrt{3}/2$ | $\sqrt{3}/2$ | 0 |

   Suppose

4. Given

$$f(x) = \frac{x+2}{1+25x^2},$$

the following parts of problem 3 are concerned with the derivative of $f(x)$ at $x = 2$.

(a) (7.5 points) Compute the approximation to the first derivative of $f(x)$ at $x = 2$ using the forward difference method for $h = 10^{-11}$.

(b) (7.5 points) Write out the code list for $f(x)$.

(c) (7.5 points) Draw the computational graph corresponding to your code list.

(d) (7.5 points) Compute the first derivative of $f(x)$ at $x = 2$ using automatic differentiation in the forward mode.

(e) (7.5 points) Compute the first derivative of $f(x)$ at $x = 2$ using automatic differentiation in the reverse mode.

(f) (7.5 points) List two significant advantages of automatic differentiation over the forward difference method and two disadvantages. Justify your answer and make specific reference to $f(x)$ given above.

5. The following integral (and it's exact solution) are to be used in answering **one** (and only one) of the following three problems. That is,you are to do either 4a, 4b, or 4c. You must indicate which part you are doing for credit, otherwise, I will grade the first one that I see. We know that

$$\pi = \int_0^1 \frac{4}{1+x^2}.$$

Here we will take the "exact" value of $\pi$ to be equal to 3.14159265359.

(a) (20 points) Let the desired accuracy for the Adaptive Quadrature be $\in = 10^{-8}$. The first application of Simpson's rule is $S(0,1) = 3.1\bar{3}$... Carry out one level (iteration) for the Adaptive Quadrature using Simpson's rule.

(b) The error for the composite Trapezoidal rule is

$$-\frac{b-a}{12}h^2 f"(c).$$

1. (9 points) Compute the maximum of the second derivative in [0,1] and use this to compute an upper bound on the error for of the application of the trapezoidal rule to the given integral with h $= \frac{1}{3}$.
2. (9 points) Use the Trapezoidal rule to compute the approximate value of the given integral for h $= \frac{1}{3}$.
3. (2 points) Compute the actual absolute error and compare it to the theoretical error you computed in 4bi above.

(c) (20 points) Compute the approximation to the integral using a three point Gaussian Quadrature method. Use the following table.

| $n = 3$ | *Roots* $r_{n,i}$ | *Coefficients* $c_{n,i}$ |
|---|---|---|
| 1 | 0.7745966692 | 0.5555555555 |
| 2 | 0.0000000000 | 0.8888888889 |
| 3 | −0.7745966692 | 0.5555555555 |

6. (20 points) Let

$$A = \begin{bmatrix} 4 & 2 & -1 \\ -2 & 6 & 1 \\ 1 & -1 & 8 \end{bmatrix}$$

(a) Compute the LU decomposition for A; A= LU where L is a lower triangular matrix

$$L = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

and D is a diagonal matrix

$$U = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Operation count

7. Let

$$A\overrightarrow{x} = \begin{bmatrix} 0.780 & 0.563 \\ 0.457 & 0.330 \end{bmatrix} \overrightarrow{x} = \begin{bmatrix} 0.217 \\ 0.127 \end{bmatrix} = \overrightarrow{b}$$

The solution is $x_1 = 1.00$, $x_2 = -1.00$

(a) (10 points) Using 3-digit arithmetic, solve this system using Gaussian Elimination with partial pivoting. You should get $\tilde{x}_1 = 1.71$, $\tilde{x}_2 = -1.98$. Note that the error in these values is nearly as large as the solution itself.

(b) (10 points) Compute the residual $\overrightarrow{r}$ which is defined to be:

$$\overrightarrow{r} = \overrightarrow{b} - A\tilde{x}.$$

Note: The residual measures difference between computing back with the approximate, $A\tilde{x}$, and what it should be $\overrightarrow{b}$. Ideally, the value of the components of the residual vector will be zero or very close to it. Note that both of the components of the residual are less than $10^{-2}$. We would not expect a better solution with 3-digit arithmetic. Moreover, we depend on the residual to tell us how "good" our solution is since it **can** be computed whereas the exact solution is unknown for real problems so that we can **never** compute the absolute error.

(c) (10 points) The approximate values computed in part (a) have errors that are roughly the same magnitude as the correct solution; that is, almost no correct digits. The value of the residual in part (b) is to within three digits accuracy of what the best residual is (zero). Explain this discrepancy. **Note**: You must make specific reference to the specific matrix and right-hand-side values we have above. You might want to graph the equations associated with this matrix problem. This will give you some insight as to what is going on.