MATH 4-5794:Optimization Modeling - NOTES

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1 INTRODUCTION

See Chapter 1 of Williams - read and supplement it with what follows.

1.1 Mathematical Models

1.1.1 Models are simplifications of our environment constructed to

1. Better understand (cause/effect relationships) or analyze to gain insight into the "real world" - for example, how to trees utilize the sun?

2. Describe the mechanism or process of a particular system - for example, what is the process by which West Nile infects horses?

3. Control or prepare (as in weather models) - for example, control an airplane or the processes in an oil refinery or chemical process.

1.1.2 Types of models

1. Experimental or models - design of a prototype solar vehicle that is entered into a contest, chemistry lab experiment, wind tunnels.

2. Virtual models - for example, virtual human at UCDHSC, or the flight simulator (Stapleton Center, Denver).

3. Mathematical models - equations and relations replace the physics, dynamics, chemistry, economics, biology

1.1.3 Model functions (Fred Murphy)

1. Transformation in time - convert T to T+1, inventory, store ice from winter to summer.

2. Transformation in space/place -location A to location B - transportation ice from the South Pole to Saudi Arabia.

3. Transformation in matter/form - $\{Q_i\}$ to $\{Q_j\}$, process raw materials into products, take silver and make jewelry, take wood and make furniture.

1.1.4 Model purpose

1. Descriptive - given input conditions what are the results, a simulation of an investment strategy.
2. Predictive - determine the number of bank queues required for each day of the week and each week of the year, determine how to invest in the stock market.
3. Prescriptive - find the "best," optimize, normative models, evaluative, has a measure of performance (objective function).

1.1.5 Modeling Process I

I. Identify the problem

A. Get an understanding of the field and context from which the problem arises

1. Literature review, read, study

a. What are the essential components that need to be known

b. What approaches are fruitful

2. State/write down the problem

3. Get an understanding of the problem

a. What has been done, what is the essence of the problem and solution techniques

b. What software systems have been used to solve the problem

II. Assumptions and simplifications - recall the radiation therapy of tumors model

III. Identification of variables and units - endogenous, what is being sought, exogenous (data, parameters)

IV. Equations and relationships

V. Data - catscans

Note: How to represent that data at hand is sometimes a very difficult problem, for example, in segmentation of a medical images. Any artificial intelligent system has data representation issues, for example, if one were to set up a cryptocript or chess system.

VI. Model

VII. Solution methods and associated software acquisition (already made, developed)

VIII. Mathematical implementation

A. Prototype - laugh test

B. Production

IX. Interpretation - the why of model results

X. Application - the implementation of the model (running the oil refinery according to the solutions and interpretation, setting the linear accelerator to the angles and intensities that the model identified in actually treating a patient)

1.1.6 Modeling Process II (see [2] page 6, Figure 1-4)

1. **Understanding** - the problem
2. **Formulation** - the model
3. **Solution** - testing, validating
4. **Implementation** and **interpretation**

Note that once the modeling process has been completed, one gains insight and modifies one's model.

1.2 Optimization Models

1.2.1 Statement of the Optimization Problem

We are imposing an order, that is, we are prescribing how we measure the performance of the system.

$$\begin{aligned} \text{optimize}(\max/\min)\; z &= f(x) \text{ - imposed order is } f(x) \\ x &\in X \\ g(x) &\leq 0 \\ X &\subseteq \Re^n \\ f &: X \to \Re \\ g &: X \to \Re^m \end{aligned}$$

The functions $f(x)$ and $g(x)$ are called the *objective function* and *constraint functions* respectively.

A. Unconstrained - $X = \Re^n, m = 0$, that is, there are no functions g.

Example:

$$\begin{aligned} \min f(x) &= x(x-1) \\ \min f(x) &= [x_1, x_2] \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 8x_1 + 3x_2 \\ x_1^* &= 2.5, x_2^* = -1, f(x^*) = -11.5 \end{aligned}$$

Note:the maximum will be infinite for both of the above examples

B. Linear Programming - both f and g are linear, and X is a "box." That is,

$$\begin{aligned} z &= f(x) = \overrightarrow{c} \cdot \overrightarrow{x} = \sum_{i=1}^{n} c_i x_i \\ g(x) &= A\overrightarrow{x} - \overrightarrow{b} \leq \overrightarrow{0}, \text{ where } A_{mxn} \end{aligned}$$

Example:

$$\begin{aligned}
\max z &= 2x_1 + x_2 \\
g(x) &= \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
x &\in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} = X \\
x_1^* &= \frac{12}{7}, x_2^* = \frac{6}{7}, z^* = f(x^*) = \frac{30}{7}
\end{aligned}$$

C. Nonlinear - either f or g (or both) are nonlinear.
D. Combinatorial or integer - $X \subseteq \{-n, -(n-1), ..., 0, 1, ..., n\}$
E. Dynamic -

$$\begin{aligned}
\vec{x} &= [x_1(t), ..., x_n(t)]^T \\
\max J(u) &= \int_a^b L(\vec{x}, \vec{u}; t) dt \\
\vec{x}'(\vec{u}; t) &= f(\vec{x}, \vec{u}; t), x(\vec{u}; 0) = x_0 \\
\vec{u}(t) &\in \Omega \subset \Re^m
\end{aligned}$$

1.2.2 Terms and Notation

- **Domain** (also called policy region): X
- **Objective Function**: $z = f(\vec{x})$
- **Constraint Function**: $g(\vec{x})$
- **Neighborhood** (of a point a)**:** $N_a(\in) = \{x \text{ such that } |x - a| \leq \in\}$
- **Feasible (constraint) Region**: $F = \{\vec{x}|\ g(\vec{x}) \leq 0 \text{ and } x \in X\}$
- **Optimality Region**:

$$\begin{aligned}
X^*_{\max} &= \arg\max\{f(\vec{x})|\vec{x} \in F\} \equiv \{\vec{x}^* \in F | f(\vec{x}^*) \geq f(\vec{x}) \forall \vec{x} \in F\} \\
X^*_{\min} &= \arg\min\{f(\vec{x})|\vec{x} \in F\} \equiv \{\vec{x}^* \in F | f(\vec{x}^*) \leq f(\vec{x}) \forall \vec{x} \in F\}
\end{aligned}$$

- **Optimal Value(s)**:

$$\begin{aligned}
z^* &= f^* = \sup\{f(\vec{x})|\ \vec{x} \in F\} \text{ if the optimization is maximization} \\
z^* &= f^* = \inf\ \{f(\vec{x})|\ \vec{x} \in F\} \text{ if the optimization is minimization}
\end{aligned}$$

- **Local Minimum**

$$\vec{x}^* :\ \vec{x}^* \in \arg\min\{f(\vec{x})|\vec{x} \in F \cap N_{x^*}(\in), for some \in > 0\}$$

- **Local Maximum**

$$\vec{x}^* :\ \vec{x}^* \in \arg\max\{f(\vec{x})|\vec{x} \in F \cap N_{x^*}(\in), for some \in > 0\}$$

- **Solution (**to the optimization problem) is the optimal value or a proof that $X^* = \{\phi\}$

1.2.3 Some Facts

1. $f^*[\max, f, X, g] = -f^*[\min, -f, X, g]$
2. $X^*[\max, f, X, g] = -X^*[\min, -f, X, g]$
3. $f^*[opt, \alpha f + \beta, X, g] = \alpha f^*[opt, f, X, g] + \beta$
4. If $T : X \to Y$ has an inverse $T^{-1} : Y \to X$, then
   (a) $f^*[opt, f, X, g] = f^*[opt, fT^{-1}, Y, gT^{-1}]$
   (b) $X^*[opt, f, X, g] = T^{-1}X^*[opt, fT^{-1}, Y, gT^{-1}]$
5. Let $X \subseteq X\,'$ and $g \geq g\,'$, then
   (a) $F[opt, f, X, g] \subseteq F[opt, f, X\,', g\,']$
   (b) $f^*[\min, f, X, g] \geq f^*[\ \min,\ f, X\,', g\,']$
   (c) $f^*[\max, f, X, g] \leq f^*[\max, f, X\,', g\,']$

1.2.4 Optimization models - Mathematical Programming (adapted from [2] page 8)

1. All optimization models require a quantitative specification of *how much* of the available resources are to be allocated to each of the competing activities.
2. For each optimization model, there is implicitly/explicitly some objective(s) to be optimized.
3. Most optimization model have constraints limiting the amount of resources that can be "consumed" by competing activities as well as, possibly, additional constraints restricting the makeup of the set of activities that can be chosen.

Formally, mathematical programming seeks to determine the values of certain *decision variables* (alternatively, the levels of specified *activities*) subject to various restrictions (*constraints*) that are expressed as equalities and inequalities in terms of the decision variables. The constraints, in practice, reflect financial, availability, technological, organizational, political, etc. considerations. The values of the decision variables are determined so as to optimize (minimize or maximize) some chosen criterion (or criteria) that, when expressed in terms of the variables, is called the *objective function* (or simply the objective). The form of the objective, the constraints, and the values of the decision variable may take determine the taxonomy and the type of mathematical programming algorithm that need to be used. The optimal values of the decision variables represent the optimal plan or *program*; thus the origin of the term "mathematical programming."

1.2.5 Elements of Mathematical Modeling in Optimization

A. Purpose

1. **Prescriptive** - attempts to direct the course of action of a system, for example, crew scheduling (optimization)

2. **Descriptive** - discloses the behavior of a system, for example, if x number of hunting licenses are given out and y elk, the elk population of elk at the end of the season is z (simulation)

3. **Explanatory** - the configuration/geometry of arteries (and air sacks of the lungs) in the body is such that it minimizes energy, coping theory explaining burnout.(system dynamics)

B. Domains of applications

1. Blending
2. Production planning
3. Oil refinery management
4. Financial and economic planning
5. Personnel planning
6. Agricultural planning
7. Medicine
8. Crew scheduling - scheduling in general

C. Analysis

1. Optimality conditions
2. Representations (see syllabus)
   - a) Deterministic
   - b) Stochastic
   - c) Interval
   - d) Possibilistic
   - f) Fuzzy
3. Structures
   - a) Linear
   - b) Nonlinear
   - c) Integer
   - d) Block diagonal
   - e) Network/transportation
4. Time
   - a) Static
   - b) Dynamic
     - (i) Difference - discrete time
     - (ii) Continuous
5. Solution methods
   - a) Simplex
   - b) Gradient
   - c) Heuristic
   - d) Interior point methods
6. Duality
7. Pre-optimization analysis

a) Infeasibility
b) Redundancy
8. Post-optimization analysis - sensitivity analysis

D. Components of an Optimization Model (see H. Greenberg's optimization modeling notes)

1.Objects
a) Data
(i) Symbolic
- Unordered sets (cities, types of cow)
- Logical connections
(ii) Numeric
- Ordered sets (time)
- Scalars, fixed but can be changed (per mile cost of operating a vehicle)
- Parameters, only God can change (gravitational constant, pi, e)
- Tables
b) Decisions
(i) Variables
- Continuous
- Discrete
(ii) Functionals
- Analytic
- Logical

2. Relations
a) Constraints
(i) Arithmetic
- Equations
- Inequalities
(ii) Logical
b) Conditionals, Meta-commands
(i) Generation
(ii) Admissibility
c) Learning

1.3 Examples

1.3.1 Greenberg Examples

The following examples are taken from [1]

1. Capital Investment

An investor has five prospective opportunities. S/he estimates returns for each to be 1, 2, ..., and N=5 times for every dollar invested in option 1, 2, ..., and N=5. The total amount that the person has to invest is b=$100,000. What should be the investment policy to maximize the sum total return?

Model: Let x_j denote the amount invested in the jth in prospect. The problem is:

$$\max z = \sum_{j=1}^{N=5} c_j x_j = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5, \text{ where } c_j = j \quad (1)$$

$$\sum_{j=1}^{N=5} x_j = x_1 + x_2 + x_3 + x_4 + x_5 \leq b = 100,000 \quad (2)$$

$$x_j \geq 0, \quad j = 1, ..., N = 5 \quad (3)$$

The objective function is (1) and is linear. Constraint (2) (linear) is the capital limitation. Constraint (3) (linear) says that negative investments are not permitted. While certain economic models allow deficit spending, this does not. Without introducing further constraints, if (3) were dropped (for any j), there could be infinite return. For example, if x_1 were allowed to be negative, consider $x_1 = -D$ and $x_2 = D$, $x_j = 0$ for $j = 3, 4, 5$. Observe that (2) is satisfied and in fact,

$$\sum_{j=1}^{5} x_j = 0,$$

so there is no investment (the available capital $100,000 is not used). Moreover, the return is $-D + 2D$ or simply D dollars. By letting D be arbitrarily large, we can receive arbitrarily large returns.

The generalization of this problem is to let b, c$_j$, and N be arbitrary in the above model.

2. Production

A manufacturer know that s/he must supply 5, 6, 8 and 6 items respectively in the next four months. The production cost of one item is 1, 4, 2 and 4 units (dollars), respectively during these months. There is no limitation on output. Even so, it might be preferable not to produce the items during the first month when production is cheapest due to the further condition that there is a storage cost of 1 unit (dollars) per month per item, and also due to the fact that production in each month must equal to or exceed that of the previous month. An increase in production costs 2 units (dollars) per item. How much should the manufacturer produce in each month?

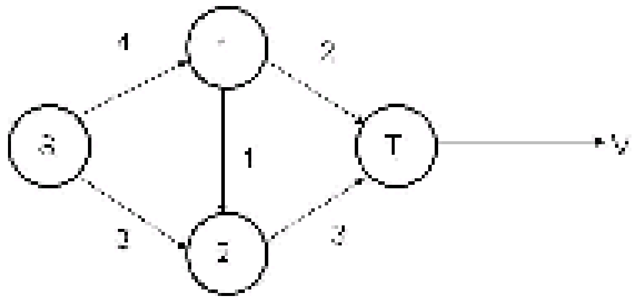
Model: Let x_j denote the amount produced in each month $j (j = 1, ..., 4)$.

The requirement that the production cannot decline is given by:

$$x_4 \geq x_3 \geq x_2 \geq x_1 \geq 0 \quad (4)$$

The demands are given by:

$$\begin{aligned} x_1 &\geq 5 \text{ (first month)}, \quad (5) \\ x_1 + x_2 - 5 &\geq 6 \text{ (second month)}, \\ x_1 + x_2 + x_3 - 5 - 6 &\geq 8 \text{ (third month)}, \\ x_1 + x_2 + x_3 + x_4 - 5 - 6 - 8 &\geq 6 \text{ (fourth month)}. \end{aligned}$$



The production cost is given by:

$$x_1 + 4x_2 + 2x_3 + 4x_4.$$

The storage cost is given by:

$$(x_1 - 5) + (x_1 + x_2 - 11) + (x_1 + x_2 + x_3 - 19) + (x_1 + x_2 + x_3 + x_4 - 25).$$

The cost due to increases in production is given by:

$$2(x_2 - x_1) + 2(x_3 - x_2) + 2(x_4 - x_3).$$

Adding the costs up, we obtain

$$3x_1 + 7x_2 + 4x_3 + 7x_4 - 60,$$

so that our linear programming problem is:

$$\begin{aligned} \min z &= 3x_1 + 7x_2 + 4x_3 + 7x_4 \\ &\quad \text{subject to (4), and (5)} \end{aligned}$$

3. Maximum Flow

An oil company wishes to pipe oil from a source, S, to a "sink," T, through a network of pipes shown below. Each link between two junctions, say (i, j), has capacity c_{ij}. What is the maximal flow from S to T?

Model: Let V denote the total flow into T, and let x_{ij} = flow from link i to link j. Then we have:

$$\begin{aligned}
x_{S1} &\leq 4 \\
x_{1T} &\leq 2 \\
x_{S2} &\leq 3 \\
x_{12} &\leq 1 \\
x_{2S} &\leq 3 \\
x_{S1} - x_{1T} - x_{12} &= 0 \\
x_{S2} + x_{12} - x_{2T} &= 0 \\
x_{1T} + x_{2T} - V &= 0 \\
x_{ij} &\geq 0, \text{ for all } i, j.
\end{aligned}$$

The first five constraints limit capacity. The next three constraints are balance equations, flow in = flow out. The generalization of this example is (we let the indices $S = 0$ and $T = N$:

$$\begin{aligned}
0 &\leq x_{ij} \leq c_{ij},\ 0 \leq i \leq N-1, 1 \leq j \leq N \\
\sum_{j=1}^{N} x_{pj} - \sum_{i=0}^{N-1} x_{iq} &= 0,\quad 0 \leq p \leq N-1, 1 \leq q \leq N \\
\sum_{j=1}^{N-1} x_{0j} &= \sum_{i=1}^{N-1} x_{iN} = V
\end{aligned}$$

4. Transportation

Suppose a need arises for buses to transport Bronco fans. In particular, at points A, B, C, and D there is a need for 3, 3, 4, and 5 buses, respectively. These 15 buses must come from garages G_1, G_2, and G_3. G_1 has 2 buses, G_2 has 6 buses, and G_1 has 7 buses, so 15 buses are available. The time to drive a bus from each garage to each destination is given in the table below. How do we distribute the buses so as to minimize total bus time?

Time in minutes

FROM/TO	A	B	C	D
G_1	13	11	15	20
G_2	17	14	12	13
G_3	18	18	15	12

Model: Let the number of buses sent from G_i to A, B, C and D be denoted x_{i1}, x_{i2}, x_{i3}, and x_{i4}, respectively. We have the following constraints:

Availability -

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &\leq 2 \\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 6 \\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 7
\end{aligned}$$

Demand -

$$\begin{aligned} x_{11} + x_{21} + x_{31} &\geq 3 \\ x_{12} + x_{22} + x_{32} &\geq 3 \\ x_{13} + x_{23} + x_{33} &\geq 4 \\ x_{14} + x_{24} + x_{34} &\geq 5 \end{aligned}$$

Objective function -

$$\begin{aligned} \min z &= 13x_{11} + 11x_{12} + 15x_{13} + 20x_{14} + \\ &\quad 17x_{21} + 14x_{22} + 12x_{23} + 13x_{24} + \\ &\quad 18x_{31} + 18x_{32} + 15x_{33} + 12x_{34} \end{aligned}$$

Note: x_{ij} must be a non-negative integer.

General Problem: Given m supply stations and n destinations, we seek to meet each demand, b_j for $j = 1, ..., n$, at each destination from our available supply a_i for $i = 1, ...m$, at each supply station. The unit cost (time, money, distance, etc.) to transport a unit of item from i to j is $c_{ij} \geq 0$. The problem is to find the least cost transportation policy. Let x_{ij} denote the amount transported from supply depot i to demand destination j. The general model is:

$$\begin{aligned} \min z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{supply side} \sum_{j=1}^{n} x_{ij} &\leq a_i, \quad i = 1, ..., m \\ \text{demand side} \sum_{i=1}^{m} x_{ij} &\geq b_j, \quad j = 1, ..., n \\ x_{ij} &\geq 0 \quad \text{for all } i, j \end{aligned}$$

5. Diet Problem - Blending

A farm wishes to supply the minimal level of nutrition to its livestock while minimizing cost. In particular, there are four nutritional ingredients (for example, calories, vitamins, minerals, protein, etc.) which we call A, B, C, and D. There are three grains that can be mixed, each containing a supply of nutrition. The data is provided in the table below

Nutrient	Grain 1	Grain 2	Grain 3	Minimal Total Requirements (per day)
A	2	3	7	≥ 1250
B	1	1	0	≥ 250
C	5	3	0	≥ 900
D	0.6	0.25	1	≥ 232.5
Cost per weight	\$41.00	\$35.00	\$96.00	

Model: Let x_j be the amount of grain j used in the diet (j = 1, 2, 3). We then have:

$$\begin{aligned} \min z &= 41x_1 + 35x_2 + 96x_3 \\ 2x_1 + 3x_2 + 7x_3 &\geq 1250 \\ x_1 + x_2 &\geq 250 \\ 5x_1 + 3x_2 &\geq 900 \\ 0.6x_1 + 0.25x_2 + x_3 &\geq 232.5 \\ x_j &\geq 0. \end{aligned}$$

General Problem: Given m nutrient requirements b_i i=1,...,m, and n foods with unit costs c_j j=1,...,n, and a matrix A, with a_{ij} = the amount of nutrient i in one unit of food j,

$$\begin{aligned} \min z &= \sum_{j=1}^{n} c_j x_j \\ Ax &\geq b \\ x &\geq 0. \end{aligned}$$

1.3.2 Lodwick Examples - Multiple Sclerosis Scab Detection by Magnetic Spin Imagining

Problem: Find the settings for two parameters (echo time and pulse repetition) of an MRI so that the resulting image distinguishes MS scabs from surrounding tissue/fluid (white matter, WM, and cranial-spinal fluid, CSF).

a. Decision variables: ET - echo time (greater than 10 but less than 200 msec)

PR - pulse repetition time (greater or equal to 100 by less than 3,000 msec)

b. Data (parameters): SD - spin density for each tissue/fluid

SL - spin lattice relaxation for each tissue/fluid

SS - spin-spin interactions for each tissue/fluid

c. Tissue fluids - MS, WM, CSF

d. Equations - spin echo, what is picked up by the MRI

$$\begin{aligned} SE_{spin-echo}(ET, PR, TF) &= SD(TF)[1 - 2\exp(-\frac{PR - \frac{ET}{2}}{SL(TF)}) + \exp(-\frac{PR}{SL(TF)})]\exp(-\frac{ET}{SS(TF)}) \\ Alpha(ET, PR) &= \alpha(ET, PR) = \frac{\sqrt{ET - 10.0}}{PR} \end{aligned}$$

	SD	SL	SS
MS	1.00	688.0	113.0
WM	0.89	370.0	70.0
CSF	1.15	2850.0	372.0

To visually detect an entity from its surrounding tissue, the two spin-echoes, SE, must be equal to or greater than, 0.004. That is,

$$\begin{aligned}
\text{MS scabs and CSF, C}_1 &: |SE_{spin-echo}(ET,PR,MS) - SE_{spin-echo}(ET,PR,CSF)| \\
&= |A_1(ET,PR) - A_2(ET,PR)| \geq 0.004 \\
\text{MS scabs and WM, C}_1 &: |SE_{spin-echo}(ET,PR,MS) - SE_{spin-echo}(ET,PR,WM)| \\
&= |A_1(ET,PR) - A_3(ET,PR)| \geq 0.004 \\
\text{WM scabs and CSF, C}_1 &: |SE_{spin-echo}(ET,PR,WM) - SE_{spin-echo}(ET,PR,CSF)| \\
&= |A_2(ET,PR) - A_3(ET,PR)| \geq 0.004
\end{aligned}$$

Model $L_1$2: - L_1-norm and two tissues

$$\begin{aligned}
\max Z_{12} &= \{\sqrt{A_1(ET,PR)^2} + \sqrt{A_2(ET,PR)^2}+\}\alpha(ET,PR) \\
\text{constraint 1} &: \sqrt{[A_1(ET,PR) - A_2(ET,PR)]^2} \geq 0.004 \\
\text{constraint 2} &: \sqrt{[A_1(ET,PR) - A_3(ET,PR)]^2} \geq 0.004
\end{aligned}$$

Note: Computationally, why is the square root of a variable squared used instead of the absolute value?

Model $L_1$3: - L_1-norm and three tissues

$$\begin{aligned}
\max Z_{13} &= \{\sqrt{A_1(ET,PR)^2} + \sqrt{A_2(ET,PR)^2} + \sqrt{A_3(ET,PR)^2}\}\alpha(ET,PR) \\
\text{constraint 1} &: \sqrt{[A_1(ET,PR) - A_2(ET,PR)]^2} \geq 0.004 \\
\text{constraint 2} &: \sqrt{[A_1(ET,PR) - A_3(ET,PR)]^2} \geq 0.004 \\
\text{constraint 3} &: \sqrt{[A_2(ET,PR) - A_3(ET,PR)]^2} \geq 0.004
\end{aligned}$$

Model $L_2$2: - L_2-norm and two tissues

$$\begin{aligned}
\max Z_{22} &= \{A_1(ET,PR)^2 + A_2(ET,PR)^2\}\alpha(ET,PR) \\
\text{constraint 1} &: [A_1(ET,PR) - A_2(ET,PR)]^2 \geq 0.004 \\
\text{constraint 2} &: [A_1(ET,PR) - A_3(ET,PR)]^2 \geq 0.004
\end{aligned}$$

Note: The actual contrast constraint yields a larger lower bound for the contrast so that the contrast will be actually higher under this model

Model $L_2$3: - L_1-norm and three tissues

$$\begin{aligned}
\max Z_{23} &= \{A_1(ET,PR)^2 + A_2(ET,PR)^2 + A_3(ET,PR)^2\}\alpha(ET,PR) \\
\text{constraint 1} &: [A_1(ET,PR) - A_2(ET,PR)]^2 \geq 0.004 \\
\text{constraint 2} &: [A_1(ET,PR) - A_3(ET,PR)]^2 \geq 0.004 \\
\text{constraint 3} &: [A_2(ET,PR) - A_3(ET,PR)]^2 \geq 0.004
\end{aligned}$$

1.3.3 Lodwick Examples 2 - Markowitz Mean/Variance Model

Financial models are often optimization models. The problem, however, is to quantify risk. Secondly how does optimize over **distributions** since distributions are the data we have? Thirdly, how does one compare when one has competing goals (maximize returns and minimize risk for example)?

Portfolio Risk and Return Suppose we have estimates for:

- Expected returns for N investments

$$\mu_n, \quad n \in \{1, \ldots, N\} = \Pi \text{ portfolio universe}$$

- Variance on the N investments

$$\sigma_n^2, \quad n \in \{1, \ldots, N\}$$

- Correlation on the N investments

$$\rho_{i,j} \quad i, j \in \{1, \ldots, N\}$$

- Standard deviation on the N investments

$$\sigma_n \quad n \in \{1, \ldots, N\}$$

- The mean of a portfolio is

$$\begin{aligned} \mu_P &= \sum_{i \in P \subseteq \Pi} w_i \mu_i \\ w_i &\geq 0, \sum_{i=1}^{N} w_1 = 1 \end{aligned}$$

  where

$$\begin{aligned} w_i &> 0 \text{ if } i \in P, \\ w_i &= 0 \text{ if } i \notin P. \end{aligned}$$

- The variance of the portfolio is

$$\begin{aligned} \sigma_P^2 &= \sum_{i \in P \subseteq \Pi} \sum_{j \in P \subseteq \Pi} w_i w_j \sigma_i \sigma_j \rho_{i,j} \\ &= \sum_{i \in P \subseteq \Pi} \sum_{j \in P \subseteq \Pi} w_i w_j \sigma_{ij} \end{aligned}$$

  where

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{i,j}$$

Markowitz in his initial model used the standard deviation, σ, as a measure of risk since it is expressed in the same units as the mean so our axes will have the same units when we plot what is called the *efficiency frontier.*

Defining Markowitz Efficiency

- A portfolio P^* is *MV efficient* if it has least risk for a given level of portfolio return. That is,

$$\mu_P = \mu_{P^*} \Rightarrow \sigma_P^2 \geq \sigma_{P^*}^2$$

- A portfolio P^* is *MV efficient* if it has maximum expected return for a given portfolio risk. That is,

$$\sigma_P^2 = \sigma_{P^*}^2 \Rightarrow \mu_P \leq \mu_{P^*}$$

Optimization Constraints

- Investing one unit of assets (money) and no negative investment (borrowing). That is,

$$w_i \geq 0, \sum_{i=1}^{N} w_1 = 1$$

- The budget condition is a linear inequality constraint. That is, the maximum amount there is to buy, \$$B$, is the constraint. We take \$$B$ as one unit. The constraint above forces us to spend the budgeted amount. We could have used a less than or equal to constraint. As it is, we are forcing ourselves to spend the entire budgeted amount.

- No short selling or deficit spending which is reflected in the nonegativity constraint.

<See chalkboard figure>

Computational Algorithms What the Markowitz model turns out to be from mathematical programming nomenclature is a *quadratic programming* problem. The constraints are those given above and these are linear. Model 1 is:

$$\begin{aligned}
\max z &= -\sigma_P^2 + 2\tau\mu_P = -\sum_{i\in P\subseteq\Pi}\sum_{j\in P\subseteq\Pi} w_i w_j \sigma_{ij} + 2\tau \sum_{i\in P\subseteq\Pi} w_i\mu_i \\
&= -w_P^T \sum w_P + w_P^T \mu_P \\
\text{subject to} &: \\
\sum_{i=1}^{N} w_1 &= 1 \\
w_i &\geq 0, \tau \geq 0 \text{ (a weight that influences how mean affects the optimum)}
\end{aligned}$$

Model 2:

$$\begin{aligned}
\min z &= \sigma_P^2 + \mu_P = \sum_{i \in P \subseteq \Pi} \sum_{j \in P \subseteq \Pi} w_i w_j \sigma_{ij} \\
\text{subject to} &: \\
w_P^T \mu &\geq \mu^* \\
1^T w &= \sum_{i=1}^{N} w_1 = 1 \\
w_i &\geq 0
\end{aligned}$$

2 INTERVAL ANALYSIS AND OPTIMIZATION

Download notes from my website.

References

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