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Introductory Notes on Possibility Theory and Fuzzy Set Theory

Professor Weldon A. Lodwick

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1. Introduction

Since we are interested in fuzzy arithmetic, the exposition is restricted to fuzzy sets as it applies to numbers. Generally speaking, of course, uncertainty means that the value of the number is not precisely known. There are different types of uncertainty associated with a parameter (that takes on numerical value(s)) depending on what is known about the parameter. It can be real (no uncertainty), interval (the parameter's value lies in an interval), stochastic (the parameter is characterized by a probability distribution) or fuzzy (the parameter is characterized by a membership function - to be defined and described below) and possibilistic (the parameter is characterized by a possibility function - to be defined below). We will see that a fuzzy membership function can be derived from a possibility distribution and a possibility distribution can be derived from a membership function. So in this sense, fuzzy numbers and numbers that possess possibility distributions are equivalent. Some authors (see for example [**?**]) classify uncertainty as vagueness, ambiguity or contradiction. These are usually applied to fuzzy entities that are used in fuzzy logic, linguistic computations, and so on. However, they have a meaning associated with fuzzy numbers and relations. That is, they have distinct meanings and relevance to numbers and relations, but we will not make use of this type of classification here. So, what we need to do is to just define a membership function and a possibility function. So let me start by outlining and restating some things that are in Dubois and Prade (see [**?**] pages 5-12) and give some extra comments. The way we'll start is to obtain fuzzy sets from possibility distributions. Possibility distributions are derived from possibility measures. So we begin from possibility measures.

Definition 1.1. *A possibility measure,* Π*, is a set-function defined on the set of subsets of* $\Omega = \{$ *real numbers* $\}$ *in our case, assuming values in the unit interval [0,1], and satisfying the following axioms:*

$$\begin{aligned}(i)\Pi(\emptyset) &= 0;\\ (ii)\Pi(\Omega) &= 1;\\ (iii)\forall I, \forall A_i &\subseteq \Omega, i \in I, \Pi(\cup_{i\in I} A_i) = \sup\{\Pi(A_i)|\ i \in I).\end{aligned}$$

Note: The name **possibility measure** may be understood from the following example which defines a set-function satisfying (i)-(iii). Let E be any set of Ω, the set function Π_E such that:

$$\forall A \subseteq \Omega, \Pi_E(A) = \{ \begin{array}{c} 1 \text{ if and only if } A \cap E \neq \emptyset \\ 0 \text{ otherwise} \end{array} \tag{1.1}$$

is a possibility measure. If E represents some event one is sure of, then $\Pi_E(A) = 1$ clearly means that event is possible. When the range of Π is no longer {0,1} as in the above, but [0,1], we can define a counterpart of the sure event E which characterizes Π, as follows:

$$\forall \omega \in \Omega, \mu_E(\omega) = \Pi(\{\omega\}) \equiv \pi(\omega); \tag{1.2}$$

μ_E is a generalized characteristic function called a *membership function* and defines what Zadeh (see [**?**]) called a fuzzy set. The right side of the above equation is the definition of a possibility *distribution*. The middle is the possibility measure of a singleton set and the left side is the membership function. Note that the fuzzy set is E and it is characterized by a membership function μ_E whose range is in [0,1]. In classical set theory, the membership function of a set E is the characteristic function, that is, $\mu_E \in \{0, 1\}, 0$ means it's out of the set, and 1 means that it is in the set. Either an element is in a set or it is not (law of the excluded middle, etc). In fuzzy set theory, an element can be in or out or any grade in-between (like the color green referring to a shirt or eyes, say). In interval analysis, one might think of an element being in or out or I don't know. That is, if one takes the statement $0 \leq [1, 2]$, the answer is "yes." The statement $0 \leq [-2, -1]$, the answer is "no." If one takes the statement (for example, trying to branch after a variable has an underflow) $0 \leq [-\frac{1}{2} \in_{machine}, \frac{1}{2} \in_{machine}]$, $\in_{machine} > 0$, then the answer is "maybe." I do not know how to characterize this "logic" and the mathematics that it determines. However, Didier Dubois seems to think it is a **deterministic** three-valued Boolean structure. That it is a three-valued structure, I agree. That it is deterministic (known and has no uncertainty), I do not agree, but I don't know how to think about it and show that interval analysis does indeed capture an uncertainty that is associated with a three valued logic. Regardless, in the above, $\pi(\omega)$ is called a *possibility distribution.* The distribution works on **elements** of the set. The membership function works on elements (points or real numbers in our case). The possibility measure works on subsets; it is a set function. Of course, the elements of the set are part of the set. On the other hand, we can create the set-function Π (the possibility **measure**) from the knowledge of μ_E (the membership function associated with a fuzzy set that works on elements of the universal set) since

$$\forall A, \Pi(A) = \sup\{\mu_E(\omega) | \omega \in A\} \tag{1.3}$$

is a possibility measure; that is, Π satisfies axioms (i)-(iii). This is why μ_E can be viewed as a possibility distribution π. When there exists an element ω such that $\mu_E(\omega) = 1$, the set E is said to be a *normalized* fuzzy set. The height of a fuzzy set is the largest value that the membership function takes on.

Remark 1. *Most text (and papers) begin by saying a fuzzy set is defined from collection of elements in the universal set Ω. Each element of Ω has a grade of belonging attached it, 0 meaning that the element does not belong to the set (with certainty) and 1 meaning that the element does in fact belong to the set. An element with gradation value of belonging between 0 and 1 means that it belongs to the set with that degree. For example, in the set of mathematicians we may have tall, medium and short mathematicians. A mathematician that is 6 feet tall may be tall to the degree 0.9 in the tall category, 0.7 in the medium category and perhaps 0.01 in the short category. A mathematician that is 8 feet tall has a 1.0 membership gradation in the tall category and .01 in the medium and a 0.0 in the short. And so on. This is the usual way that fuzzy sets is presented. What I think is most useful in Dubois and Prade's approach is that they begin with a measure. From a measure they define a distribution, from a distribution they define the membership (gradation) function associated with a fuzzy entity. Thus, they begin with first principles. Note that in the fuzzy set describing mathematicians, a person can simultaneously belong to several distinct sets at once. That is, usually a person that is short is not tall. However, a person that is 5'6" may be both tall and short (to a degree of 0.5 say) at the same time. Another example is a crowd of people composed of many students that have had a lot of beer in Columbus, Ohio that gathers to celebrate Ohio State University's victory over University of Michigan in football (on High Street) may be both in a state of riot and not riot at the same time. Thus, fuzzy set theory (and possibility theory) can capture uncertainties and contradictions and ambiguities and mathematically manipulate these ideas in consistent, meaningful and useful ways.*

Remark 2. *One of the open questions for me is how to manipulate fuzzy algorithms that use real numbers in such a way that one "contracts" (ala the contraction mapping theorem). That is, one may start out uncertain in a fuzzy domain and by applying an algorithm, one contracts to an interval, say.*

Remark 3. *Axiom (iii) means that* $\max\{\Pi(A), \Pi(\overline{A})\} = 1$, *where* $\overline{A}$ *is the set complementation of the set* A. *Moreover, and this is a key difference between fuzzy set theory and probability theory (see* [**?**] *for an extended discussion on the similarities and differences),* $\Pi(\overline{A})$ *cannot always be computed from* $\Pi(A)$. *That is, in probability theory we have* $p(\overline{A}) = 1 - p(A)$, *since probability measures are additive and possess the law of the excluded middle -* $p(\Omega) = 1 = p(A \cup \overline{A}) = p(A) + p(\overline{A})$. *For possibility measures, we only have the supremum holding; that is,* $\Pi(\Omega) = 1 = \Pi(A \cup \overline{A}) = \sup\{\Pi(A), \Pi(\overline{A})\}$. *Therefore, the underlying structure is "weaker" (more general) and definitely different than probability theory.*

Remark 4. *Let me say as an aside that there are many probability theorists who do not agree that possibility theory is needed nor do they think that it is different. Their approach is that anything that can be done in possibility theory (fuzzy set theory) can be done in probability theory. Fuzzy set theory and possibility theory (its underlying mathematical structure) are a young theory (circa 1965) and so its usefulness is still being debated. Its biggest successes have been in fuzzy logic controllers (engineered into video cameras, subways, chemical processes associated with steel production, etc) and AI systems (Didier Dubois' institute in Toulouse France for example). I cannot agree with the probabilists who think there is no difference and those that say even if there is a difference, fuzzy set theory is useless, trivial, etc. Fundamentally, fuzzy set theory relaxes the Aristotelian rule of the excluded middle (that comes down to us in classical set theory) that says an element either belongs to a set or (exclusive or) it does not. So fuzzy set theory is like non-Euclidean geometry. Moreover, possibility theory (that underlies fuzzy set theory) does capture uncertainties that probability theory does not and cannot capture just like non-Euclidean geometry captures structures that cannot be captured by Euclidean geometry. I really don't understand why there is a controversy.*

There is a companion set-function to possibility called the *necessity* N defined by:

$$N(A) = 1 - \Pi(\overline{A}) \tag{1.4}$$

and it satisfies axioms (i) and (ii) and the dual of (iii), namely:

$$(iv) N(\cap_{i \in I} A_i) = \inf\{N(A_i) \mid \ i \in I\}.$$

N is called the *necessity measure.* Necessity of an event can be thought of measuring the degree of impossibility of the opposite event. From the properties,

$$\Pi(A) \ < \ 1 \Longrightarrow N(A) = 0, \ \textit{and} \tag{1.5}$$

$$\Pi(A) \quad \geq \quad N(A). \tag{1.6}$$

Moreover, one can create a necessity measure from a membership function μ_E as follows:

$$\forall A, N(A) = \inf\{1 - \mu_E(\omega) | \ \omega \in A\}. \tag{1.7}$$

And of course, one can create a *necessity distribution* from a necessity measure as was done for possibility. Dubois and Prade further state (see [**?**] page 6), "Possibility measures differ strongly from probability measure in their axioms and properties, hence in their interpretations. In particular, probability measures assuming values in {0,1} focus on points (i.e., are Dirac measures), while ... possibility and necessity measures focus on sets. More generally, probability measures model precise but scattered pieces of information while possibility measures model an imprecise but consistent body of knowledge, viewed as a weighted family of nested subsets of Ω." Here the nested sets are the $level - cuts$ or as some authors call them, $\alpha - cuts$.

(see [**?**] page 6) A *level-cut* or an $\alpha - cut$ is the set

$$E_\alpha = \{x \mid \mu_E(x) \geq \alpha\}. \tag{1.8}$$

Thus, a fuzzy set can be considered as the set:

$$\{E_\alpha \mid \alpha \in (0, 1]\}. \tag{1.9}$$

Here I think they mean for the fuzzy set E to be:

$$E = \{(E_\alpha, \alpha) \mid \alpha \in (0, 1]\}. \tag{1.10}$$

This, **??**, is a set in $\Re^2$ rather than a set, **??**, in $\Re^1$. Otherwise, a fuzzy set as defined by **??** is just an interval (in the case of fuzzy numbers - see below) or a set of disconnected intervals for membership functions that are multi-modal (have more than one "hump"). I have not asked Dubois and Prade about this, but they need to be asked.

There are many properties associated with membership functions, including relations; e.g., equality, subsets of fuzzy sets, that can be defined. I will skip these for the moment. They can be found in the references. For our purpose, we are interested in numbers and this restriction to numbers will result in these alpha cuts being intervals. I will say that one of the properties that is crucial from an interval analysis point of view is one can uniquely define a fuzzy set by its alpha cuts and a fuzzy set uniquely defines alpha cuts. Thus, we can derive fuzzy sets from nested intervals and so interval analysis is a "subset" of fuzzy interval analysis (see below and the papers cited and the papers I have sent you).

Figure 1.1:

Definition 1.2. *(see [?] page 8) A fuzzy quantity, Q, is a fuzzy set of the real line $\Re$.*

μ_Q is interpreted as the possibility distribution on the values which a variable X may assume (with respect to Q).

Example 1.3. *(see figure 1.1) A fuzzy $Q = 0.5$ might be described by a fuzzy membership function $\mu_{0.5}(x) = -4x(x-1), 0 \leq x \leq 1$.*

The $0 - cut$ is the *support* and in this case is [0,1]. The $\frac{1}{2}$-cut is $[\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2} + \frac{\sqrt{2}}{4}]$. The $1 - cut$ is [0.5,0.5] and there are various definition for this point (unique in the case of fuzzy numbers). The Max of the membership function is called the *height*. The points for which the membership function is 1 are call the *modal values* of the fuzzy set. There is a special kind of fuzzy quantity that extends the concept of interval, called a *fuzzy interval*.

Definition 1.4. *(see [?] page 8) Note: I am modifying their definition, since for a mathematician, what they have is a bit confusing to me. A fuzzy interval M is a fuzzy quantity with a quasi-concave membership function (a piecewise concave function is what I think they mean). If you take a fuzzy set as defined by* **??**, *the points under and on the quasi-concave membership function bounded by the x-axis, rather than* **??**, *you will obtain a convex fuzzy set or* $\Re^2$. *Dubois and Prade say it is a convex set in* $\Re$ *on page 8. Well, it is, since it's an interval the way define it. What they call a convex fuzzy set is a fuzzy set determined by the membership function with the following property:*

$$\forall x, y, \forall z \in [x, y], \mu_M(z) \geq \min\{\mu_M(x), \mu_M(y)\}. \tag{1.11}$$

This is their way of defining what is a quasi-concave (membership) function. The resultant fuzzy set (as defined by **??**), whose membership function is **??**, a convex set. I would start out with the definition of quasi-concave by **??** and then say that a fuzzy interval has a quasi-concave membership function. In essence, this is what they do.

Definition 1.5. *A fuzzy number M is a fuzzy interval with bounded support upper semi-continuous membership function with unique modal value m (m = 0.5 in the above example).*

I believe that quasi-concave functions are upper semi-continuous ... I forget all the theorems from real and functional analysis. I will try to find out. What is important about all of this is that the level sets of fuzzy numbers are connected, closed and bounded sets of real numbers; that is, intervals (we could use Hansen's and Yohi's extended interval arithmetic for unbounded cases). Therefore, fuzzy arithmetic is done on the $\alpha-$levels.

Example 1.6. *Let me give an example. Suppose we had a fuzzy number whose membership functions were triangles (linear). So, for example, a fuzzy 3 could be defined as 2/3/5. That is, the base (support) is [2,5] and the modal value is 3. So on the left, we have the line going through the points (2,0) and (3,1) and on the right side, the line goes through the points (3,1) and (5,0) (it's a "skewed" triangle). We'd have y = x - 2 for x* $\in$ *[2,3] and the right part of the membership function is* $y = -\frac{1}{2}x + \frac{5}{2}$. *Now suppose a fuzzy 4 were 1/4/5 (skewed to the left). Then fuzzy 3 + fuzzy 4 = fuzzy 7 = 3/7/10 (and the membership value is obtained from the triangle whose base is [3,10] and whose height at 7 is 1 so it is the triangle whose vertices are (3,0), (7,1), and (10,0)). Fuzzy 3 times fuzzy 7 = fuzzy 21 = 2/21/25. Division and subtraction are handled in the same way.*

Figure 1.2:

References

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