Designs, Fans and Ovoids of PG(3,q)

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Dec. 1, 2006

A Design from Hyperovals

Consider a hyperoval Ω (q+2 points, no three collinear) in a projective plane π of even order q.

There are two related designs we wish to consider:

Both have point set $\mathfrak{P} = \pi - \Omega$ of size $q^2 - 1$.

Design S(Ω) has blocks

$$s_A = \{X \in \mathfrak{P} \mid XA \text{ is a secant line of } \Omega\} \cup \{A\} \quad \forall A \in \mathfrak{P}.$$

Design E(Ω) has blocks

$$e_A = \{X \in \mathfrak{P} \mid XA \text{ is an exterior line of } \Omega\} \quad \forall A \in \mathfrak{P}.$$

E(Ω) is a symmetric $(q^2-1, \frac{1}{2}q^2, \frac{1}{4}q^2)$ design and its complementary design S(Ω) is a Hadamard

$(q^2-1, \frac{1}{2}q^2-1, \frac{1}{4}q^2 - 1)$ design.

A Design from Hyperovals

Jennifer Key attributes this construction to Shrikhande and Singh (1962), but I first saw it in a paper by Esther Seiden (1963). However, Seiden claims that she used it in a 1961 paper and notes that Shrikhande in a private communication claimed to have first used it in 1949.

Antonio Maschietti has studied the design in several recent papers and I will refer to several of his results.

One of these is:

$S(\Omega)$ is the 2-design of points and hyperplanes of a projective geometry iff π is the Desarguesian plane $PG(2,4)$.

Regular Triples

An unordered triple {X,Y,Z} of distinct points of $\mathfrak{P}$ is called ***regular*** if, for every point P of $\mathfrak{P}$ \{X,Y,Z}, at least one of the lines PX, PY, PZ is a secant of Ω.

Maschietti is interested in such triples because they correspond to ***lines*** in the design S(Ω). [Recall that a line in a design through two points is the intersection of all blocks that contain the two points.]

Note that an alternative definition for a regular triple of points {X,Y,Z} is that $e_X \cap e_Y \cap e_Z = \emptyset$.

Regular Triples

Maschietti shows:
If $\{X,Y,Z\}$ is a regular triple and $q > 4$, then X, Y and Z are collinear on a secant line.

(Note: If for every X and Y there exists a Z so that $\{X,Y,Z\}$ is a regular triple then $q = 4$.)

A secant line s to the hyperoval Ω is said to be ***strongly regular*** if for each pair of points X and Y on $s\backslash\Omega$ there exists a third point Z on $s\backslash\Omega$ such that $\{X,Y,Z\}$ is a regular triple.

A hyperoval Ω with a strongly regular secant s is said to be ***s-regular***.

s-regular Hyperovals

Theorem: If π is a translation plane of even order $q > 4$ then a hyperoval is s-regular iff it is a translation hyperoval.

Theorem: If π = PG(2,q), q even then if a hyperoval Ω is s-regular and t-regular then
1) the secant lines s and t meet at a point N on Ω,
2) Ω is m-regular for all, and only, lines m through N and
3) Ω-{N} is a conic (Ω is a hyperconic).

s-partitions

Let Ω be an s-regular hyperoval. We define the following equivalence relation on the points of $\mathfrak{P} - s$:

$$P \sim Q \text{ iff } s_P \cap s = s_Q \cap s.$$

The equivalence classes of this relation are q-arcs, and there are q-1 of them. Each of these q-arcs, together with the two points of s on Ω is a hyperoval.

The set of these q-1 hyperovals together with Ω is called an ***s-partition*** (in earlier papers Maschietti called this set a **hyperoval bundle** ... but was probably convinced that this was a poor choice.)

Properties of s-partitions

Theorem: If π is a translation plane of even order $q > 4$ then an s-partition with respect to the translation hyperoval Ω consists of translates of Ω (all the hyperovals are translation hyperovals.)

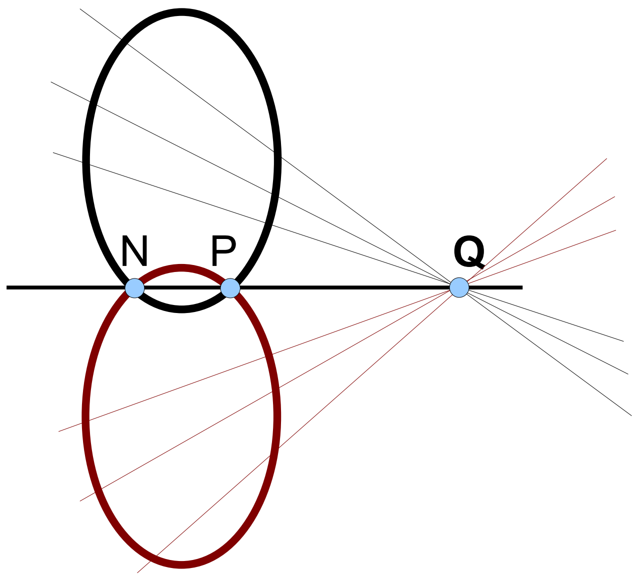
Lemma: Let Ω be an s-regular hyperoval and $P(\Omega)$ the s-partition generated by Ω. Let H be a hyperoval with $H \cap s = \Omega \cap s$. Then the following are equivalent:

(1) $H \in P(\Omega)$,

(2) for every $X \in s \backslash \Omega$, if a line on X other than s is an Ω secant and an H secant, then every Ω secant on X is an H secant,

(3) Ω and H have the same regular triples on s.

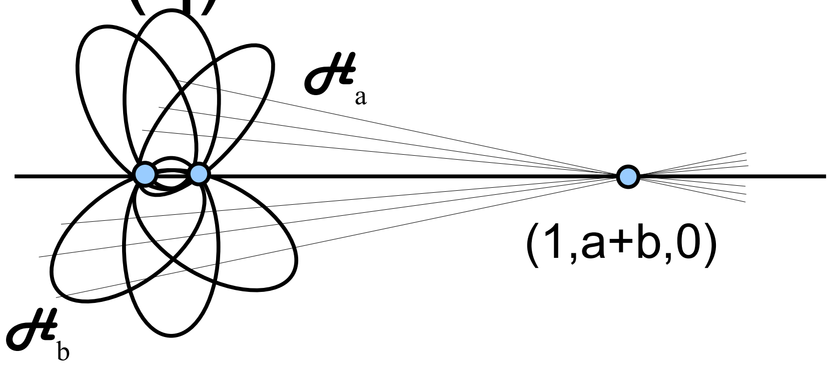
Fans of Hyperovals



Two hyperovals, $\mathcal{H}_1$ and $\mathcal{H}_2$, meeting precisely in two points N and P are said to be ***compatible*** at a point Q of the line $\mathcal{L} = \overline{NP}$, other than N or P, if all lines through Q other than $\mathcal{L}$ which are secant lines of $\mathcal{H}_1$ are exterior lines of $\mathcal{H}_2$ (and consequently, the exterior lines to $\mathcal{H}_1$ are secant lines of $\mathcal{H}_2$.)

Fans of Hyperovals

Consider a set of q hyperovals $\{\mathcal{H}_s\}$ indexed by the elements of GF(q) which mutually intersect precisely at the points (0,1,0) and (1,0,0) and for which $\mathcal{H}_a$ and $\mathcal{H}_b$ are compatible at (1,a+b,0) for all distinct a,b $\in$ GF(q).



Any set of hyperovals that are projectively equivalent to these is called a ***fan of hyperovals***.

The Plane Equivalent Theorem

Glynn (1998) & Penttila (1999):

Any ovoid of PG(3,2^h) is equivalent to a fan of hyperovals in a plane PG(2,2^h).

The proof is constructive. Starting with the secant planes of an ovoid passing through a common tangent line of the ovoid and indexed by the elements of the field, homographies, dependent on the index, map each section of the ovoid into a single secant plane of this pencil. The resulting collection of ovals have a common nucleus and form a fan of hyperovals. The process is reversible and the ovoid can be recovered from the fan of hyperovals.

The Fan Design

Given a fan F = $\{\mathcal{H}_s\}$ for each line ℓ intersecting z = 0 at a point other than (0,1,0) or (1,0,0) we can associate two sets, the set of indices of hyperovals for which ℓ is a secant line and the complementary set of indices for which ℓ is an exterior line.

With these sets as blocks we have a

$$2 - (q, \tfrac{1}{2}q, \tfrac{1}{4}q(q-2)) \text{ design}$$

called the ***Fan Design***.

Standard Fans

A special case of the Fan Design has each block repeated ½q times.

This case arises when each line through a point of $z = 0$ is a secant of the same set of $q/2$ hyperovals. A fan with this property is called a ***standard fan***.

Of the three known fans, two are standard fans (the ones where the hyperovals in the fan are translation hyperovals.)

The underlying design (no repeated blocks) of a standard fan is a

$$2 - (q, \tfrac{1}{2}q, \tfrac{1}{2}q - 1) \text{ design.}$$

Note: If $q = 2^e$ these are the parameters of $AG_{e-1}(e,2)$.

s-regular Hyperovals

One (the most difficult to establish) of many characterizations of s-regular hyperovals that Maschietti provides is based on:

Theorem: Let Ω be a hyperoval in a projective plane of order $q > 2$ and s a secant line of Ω. Let $\{M,N\} = \Omega \cap s$ and, if X and Y are distinct points on $s \setminus \{M,N\}$, let $W = e_X \cap e_Y$ and $\{Z_1, \ldots, Z_{q-3}\}$ be the set of remaining points on s. If

(1) every exterior line on Z_k, $k = 1,\ldots,q-3$ has a constant number a_k of points of W; and

(2) every line other than s on M or N has $q/4$ points of W,

then there exists a point $Z \in \{Z_1,\ldots,Z_{q-3}\}$ such that $\{X,Y,Z\}$ is a regular triple.

A standard fan is an s-partition

Theorem: A standard fan is an s-partition.

Let Ω be a hyperoval in a standard fan. We first show that Ω is s-regular with respect to the secant s: {z = 0} using the previous theorem.

Let X and Y be any two points of s\Ω. W consists of $\frac{1}{4}q^2$ points that are on exterior lines of Ω through X and Y. W is the disjoint union of q/4 q-arcs which are the q/4 hyperovals of the fan compatible with Ω at both X and Y (minus their common points). Let Z be any other point of s\Ω. Suppose that an exterior line of Ω meets W in a_z points. This line is a secant line of $\frac{1}{2}a_z$ of the q-arcs which compose W. Since the fan is standard, every exterior line through Z meets these same q-arcs and hence W in a_z points.

A standard fan is an s-partition

Theorem: A standard fan is an s-partition.

As all of the q/4 hyperovals forming W contain the common points, each line through either of the points meets W in q/4 points.

Thus, for each pair X,Y of points of s\Ω there is a regular triple containing them, so Ω is an s-regular hyperoval.

Now, consider the s-partition determined by Ω. Let H be any hyperoval in the fan. For any point X of s\Ω, consider a secant line of Ω through X which is also a secant line of H. Since the fan is standard, every secant of Ω through X will be a secant of H. By the Lemma (2) H is in the s-partition. ❑

Standard Fan $\equiv$ Classical Ovoid

In a Desarguesian plane, an s-partition consists of translation hyperovals. Therefore, a standard fan consists of translation hyperovals and the ovoid corresponding to this fan will contain a pencil of translation ovals.

By a result of O'Keefe and Penttila, any ovoid of PG(3,q) with a pencil of translation ovals is classical – either an elliptic quadric or a Tits ovoid.

Note: Any translation hyperoval gives rise to an s-partition, but only conics and one special type of translation hyperoval can occur in a standard fan. Thus, not all s-partitions are fans which implies that the assignment of field elements to hyperovals to obtain a fan is not possible for general s-partitions.

Non-standard Fans

The classification of ovoids in PG(3,q) is now reduced to the case of studying non-standard fans.

There is only one example of such a fan, which corresponds to the Tits ovoid.

Note: With $q = 8$, the fan design for this non-standard fan is the complement in the combinatorial design of all 4-sets of a an 8-set of the Hadamard 3-design which is the one point extension of the Fano plane.

References

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