The Ubiquitous Translation Hyperoval Revisited

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Normfest – March 28, 2009

Translation Hyperovals

A *translation oval Ω with axis ℓ* in a projective plane π of order n is an oval with tangent line ℓ such that the affine plane π\ℓ admits a translation group T having the affine points of Ω as a T-orbit.

- $n = 2^e$
- T is an elementary abelian 2-group of order n
- every involution of T has a different center
- there is a unique point N on ℓ\Ω which is not the center of an involution in T.

Translation Hyperovals

$\Omega \cup \{N\}$ is called a *translation hyperoval* with axis ℓ.

If P is the point of tangency, then {P, N} is called the *carrier set* of the translation hyperoval.

Note that $\{\Omega - \{P\}\} \cup \{N\}$ is also a translation oval.

In PG(2,2^e), Ω and this other translation oval are not projectively equivalent.

"This is too fussy a definition of equivalence"

Translation Hyperovals

A translation oval having more than one axis is a conic, and all the tangent lines are axes.

A conic together with it's nucleus is called a hyperconic.

Note that each “pointed conic” (remove a point of the conic and add the nucleus) is a translation oval which is not a conic.

A translation oval not contained in a hyperconic is called proper.

The Ubiquity Papers

Jha, Johnson “On the Ubiquity of Denniston-type Translation Ovals in Generalized André Planes”, Annals of Discrete Math (1992)

Jha, Johnson “A Characterisation of Spreads Ovally-Derived from Desarguesian Spreads”, Combinatorica (1994)

The Ubiquity Papers

Let S be a spread in $\mathscr{P}$ = PG(2N-1,2), consisting of $2^N + 1$ pairwise skew subspaces, each of dimension N – 1. An N-1 dimensional subspace τ of $\mathscr{P}$ is a transversal to S, relative to the carrier set $\{X,Y\} \subset S$, if

$$|\gamma \cap \tau| = 1, \ \forall\, \gamma \in S - \{X,Y\},$$
$$|\gamma \cap \tau| = 0 \text{ if } \tau \in \{X,Y\}.$$

Thm: The translation plane π_S, associated with S, admits a translation oval relative to the natural axis precisely when S admits a transversal.

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This can be rephrased in terms of spread sets.

A collection τ^* of $2^N + 1$ distinct $N \times N$ matrices over $\mathbb{F}_2$ is a translation spread set if $\tau^* = \tau \cup \{A\}$ is such that:

a) τ is an $\mathbb{F}_2$ spread set; i.e., $O, I \in \tau$ and the difference between any two distinct elements of τ is non-singular; and

b) A is a non-singular matrix such that

$$\text{rank}[A-M] = 1, \forall M \in \tau.$$

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Consider the N-1 dimensional subspace given by

$$\theta_M = \{(x, x^r) : x \in \mathbb{F}_2^N, r = 2^M\}.$$

If θ_M is a transversal of a spread S, then θ_M is called a λ-conic (with carrier set {X,Y}, the X and Y -axis of S).

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A transversal of a Desarguesian spread which remains a transversal of a spread obtained from the Desarguesian spread by replacement, gives rise to a translation oval of the new plane which is said to be inherited from the Desarguesian plane.

Most examples of translation ovals in non-Desarguesian planes are of this inherited type.

λ-conics are inherited.

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Several results are obtained concerning when generalized André spreads admit λ-conics.

Some of these in particular,

- If N is a prime power every generalized André spread admits λ-conics.
- If N = a non-prime there exist non-Desarguesian André spreads which admit λ-conics.
- If N is even, but not a power of 2, there also exist André spreads which do not admit λ-conics.

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A characterization of which generalized André planes admit λ-conics is also given.

A transversal cover of a spread S in $\mathscr{P}$ = PG(2N-1,2), relative to a pair of distinct components {X,Y}, is a collection Θ of pairwise skew transversals to S such that:

a) Each transversal in Θ is skew to both X and Y;

b) $\bigcup \Theta = \bigcup \Sigma$, where Σ = S - {X,Y}.

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A transversal cover Θ of S, when it exists, is a replacement for the partial spread $\Sigma \subset S$, and we say that the replaced spread $T = \Theta \cup \{X,Y\}$ is ovally derived from S.

Thm: A spread is ovally derived from a Desarguesian spread if and only if it is a generalized André spread that admits λ -conics.

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Subsequently it is shown that:

A nearfield plane is ovally derived from a Desarguesian plane if and only if it arises from a “strong Dickson” pair.

A Hall plane of even order q^2 is ovally derived from a Desarguesian plane if and only if q is a square.

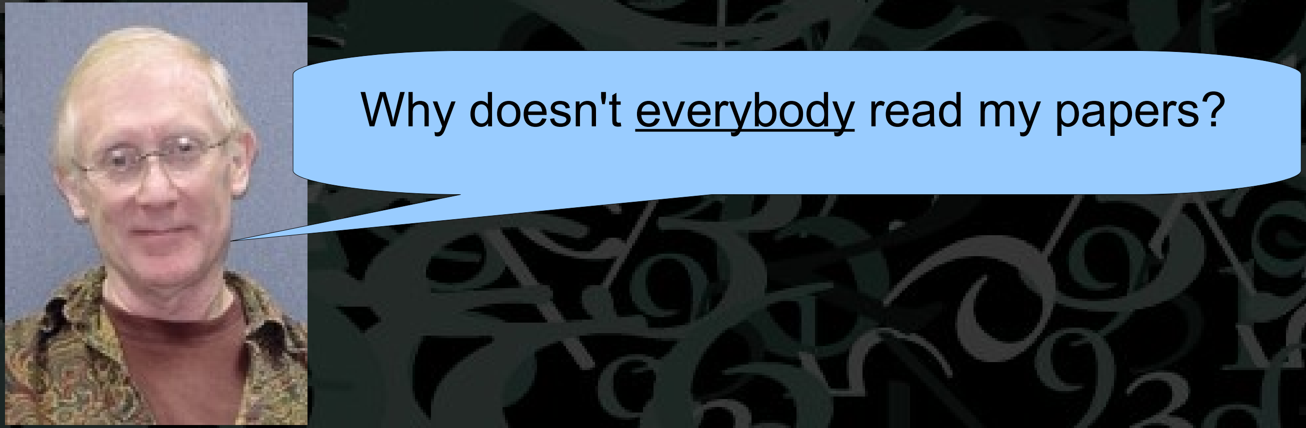
λ-conics

- Denniston (1979)

  Demonstrated the existence of translation ovals in non-Desarguesian planes by constructing some λ-conics (with M = 1) in some André planes.

- Glynn & Steinke (1993)

  Proved the existence of λ-conics (with M = 1) in even order Hall planes, H(q^2), when q is a square.



Hall Planes

A Hall plane $H(q^2)$ is constructed from $PG(2,q^2)$ by a process called ***derivation***. (A special type of subspread replacement).

This requires a ***derivation set*** D – a set of q+1 points on ℓ_∞ such that the set of all subplanes of order q (***Baer subplanes***) that contain D has the property that for any two points of the affine plane that are on a line with slope in D are contained in one of these subplanes.

Hall Planes

To obtain the Hall plane with respect to a derivation set D, we remove all the lines of PG(2,q^2) which intersect D (this includes ℓ_∞) and declare all the Baer subplanes which belong to D to be lines. This produces an affine plane which we extend to a projective plane in the usual way. The resulting Hall plane is non-Desarguesian iff $q > 2$.

Note: Collineations of PG(2,q^2) which preserve D will also be collineations of H(q^2).

Other Translation Hyperovals

Thm: Any translation hyperoval with axis the infinite line of a Hall plane of even order having one, and only one, point of it's carrier set in the derivation set, is an inherited translation hyperoval.

- Crismale (1981) – produced some examples.
- Korchmáros (1986) – gave examples of hyperconics of this type.
- O'Keefe, Pascasio, Penttila (1992) – established the result for all hyperconics of this type.
- WEC (unpublished) – extended the result to all translation hyperovals.

Classification of Hyperconics

Thm: In Hall planes of even order, $H(q^2)$, the only inherited hyperconics are those with carrier set on the infinite line, with one and only one of the carrier points in the derivation set and if q is a square, the Glynn-Steinke hyperconics (λ-conics).

- O'Keefe & Pascasio (1996) – started this classification by showing that hyperconics with both carrier points in the derivation set do not inherit.
- WEC (2009?) - finished the classification by showing that there are no inherited hyperconics in the remaining cases.

Hall(16)

A complete search for hyperovals in Hall(16) was made in 1991, but the table reporting the results was incorrect.

Hyperovals of Hall(16) containing the origin

Type	Class	Number	Group
H1 (10)	---------- None -----------		
H2 (60)	I	15	T sd Z_4
	II	15	T sd Z_4
H3 (60)	I	60	T
H4 (6)	I	30	T sd $(Z_5$ sd $Z_4)$

Hall(16)

The hyperovals in this plane:

H1 – (none) two infinite points in the derivation set.

H2 – O'Keefe-Pascasio-Penttila
the two classes correspond to which carrier point is in the derivation set.

H3 – Non-inherited

H4 – Glynn-Steinke (6 pairs of conjugate infinite points)

Non-inherited Hyperovals

We will restrict ourselves to PG(2,16), but the construction technique (alas, not the proof) most likely generalizes to higher orders. Throughout this section $q = 4$.

Let η be an element of norm 1 other than 1. ($\eta^{q+1} = 1$)

Consider two conics, having affine equations:

$\mathcal{C}_1 : Ax^2 + (\eta+1)x + y = 0$, and

$\mathcal{C}_2 : Ax^2 + (\eta+1)x + y + (\eta+1)/A = 0$,

where $A \neq 0$.

Note: These have the same point of tangency and nucleus.

Non-inherited Hyperovals

The idea is to pick $q^2/2$ affine points from each conic so that the resulting set has the appropriate structure.

Let T_0 be the set of elements of absolute trace 0 in $GF(q^2)$, and define

$$G_1 = \frac{\eta^2+\eta+1}{A} T_0 \quad \text{and} \quad G_2 = GF(q^2) \setminus G_1$$

$\mathcal{S}$ is constructed by taking the points of $\mathcal{C}_1$ whose x-coordinates lie in G_1 and the points of $\mathcal{C}_2$ whose x-coordinates lie in G_2.

Non-inherited Hyperovals

1. $\mathcal{S}$ is a translation set.
   $\mathcal{S}$ is invariant under the translations $(x,y) \to (x+a,y+b)$ for every $(a,b) \in \mathcal{S}$.
2. There are $q^2/4$ lines with slope 1 which meet $\mathcal{S}$ in 4 points and all other lines meet $\mathcal{S}$ in no more than 2 points.
3. The vertical lines and lines with slope $\eta + 1/\eta$ are all tangent lines to $\mathcal{S}$.
4. The Baer subplanes belonging to D meet $\mathcal{S}$ in no more than 2 points.
5. In Hall(16) $\mathcal{S}$ extends to a translation hyperoval.

Happy Birthday Norm!