Ovals in Hall Planes

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Hall Planes by Derivation

A Hall plane $H(q^2)$ is constructed from $PG(2,q^2)$ by a process called ***derivation***.

This requires a ***derivation set*** D – a set of q+1 points on ℓ_∞ such that the set of all subplanes of order q (***Baer subplanes***) that contain D has the property that for any two points of the affine plane that are on a line with slope in D are contained in one of these subplanes.

It can be shown that $D = \{\infty\} \cup \{(m) : m \in GF(q)\}$ is a derivation set. All derivation sets are equivalent under the group of the plane which preserves ℓ_∞.

Hall Planes by Derivation

To obtain the Hall plane with respect to a derivation set D, we remove all the lines of PG(2,q^2) which intersect D (this includes ℓ_∞) and declare all the Baer subplanes which belong to D to be lines. This produces an affine plane which we extend to a projective plane in the usual way. The resulting Hall plane is non-Desarguesian iff $q > 2$.

Note: Collineations of PG(2,q^2) which preserve D will also be collineations of H(q^2).

More on Derivation Sets

We will not use the “standard” derivation set, rather the derivation set

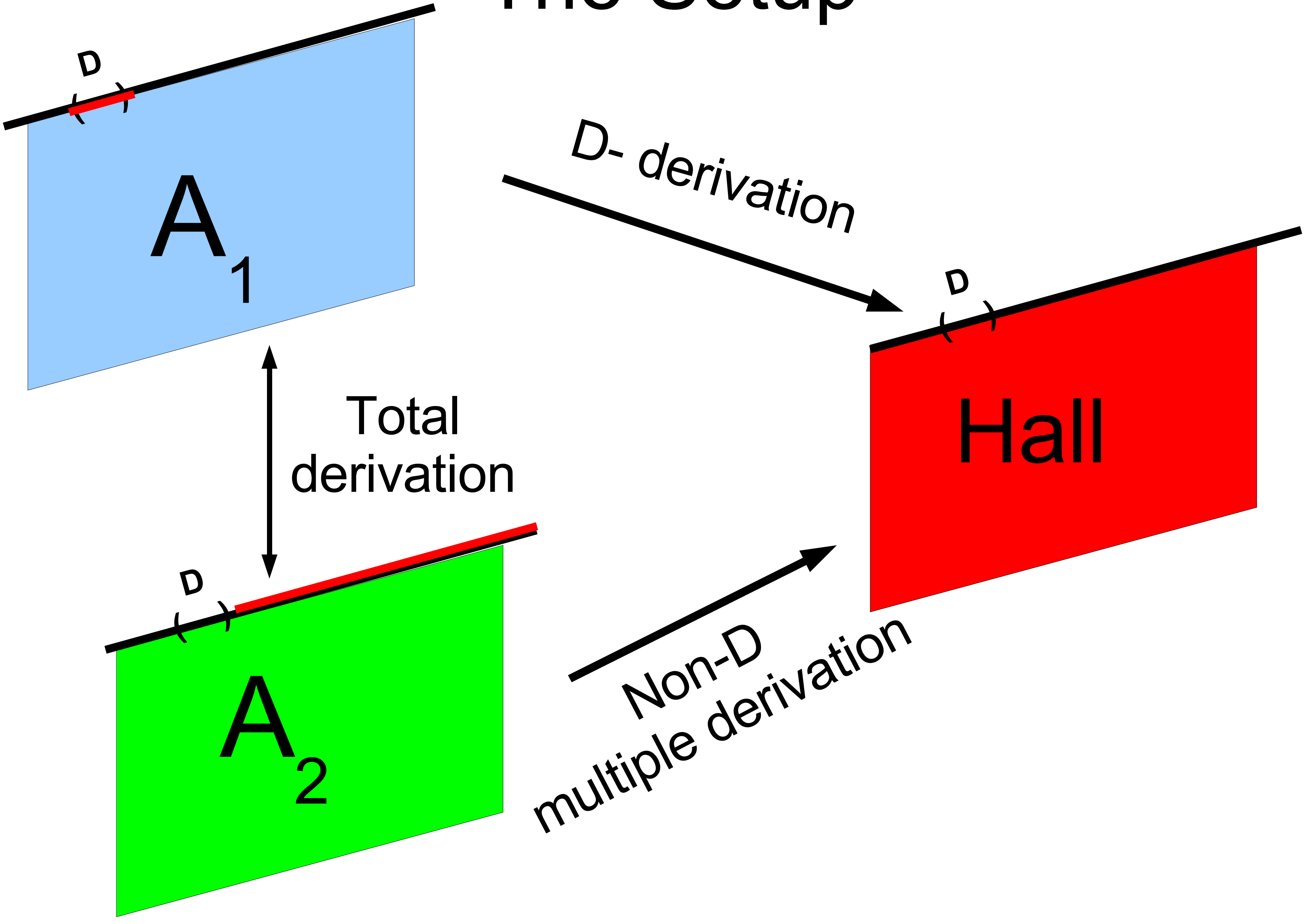
$$D = \{(n) : n^{q+1} = 1\}.$$

All the proper cosets of this multiplicative group are also derivation sets (and the elements of $GF(q)\setminus\{0\}$ can be taken as representatives if q is even).

One can independently derive with respect to any selection of these disjoint derivation sets. This is called multiple derivation and produces (generalized) André planes.

Multiple derivation using all these derivation sets gives a Desarguesian plane, and all except one gives a Hall plane.

The Setup


D
A1
D- derivation
D
Hall
Total
derivation
D
A2
Non-D
multiple derivation

Some Details

With our choice of D we have:

1) Points of Baer subplane belonging to D satisfy the equation **$y = Ax^q + B$** with $A \in D$, $B \in GF(q^2)$.

2) Total derivation between A_1 and A_2 is given by the involutory map **$(x,y) \leftrightarrow (x^q, y)$**.

Inherited Ovals

An oval in PG(2,q^2) whose point set* is an oval in the corresponding Hall plane is called an ***inherited oval***.

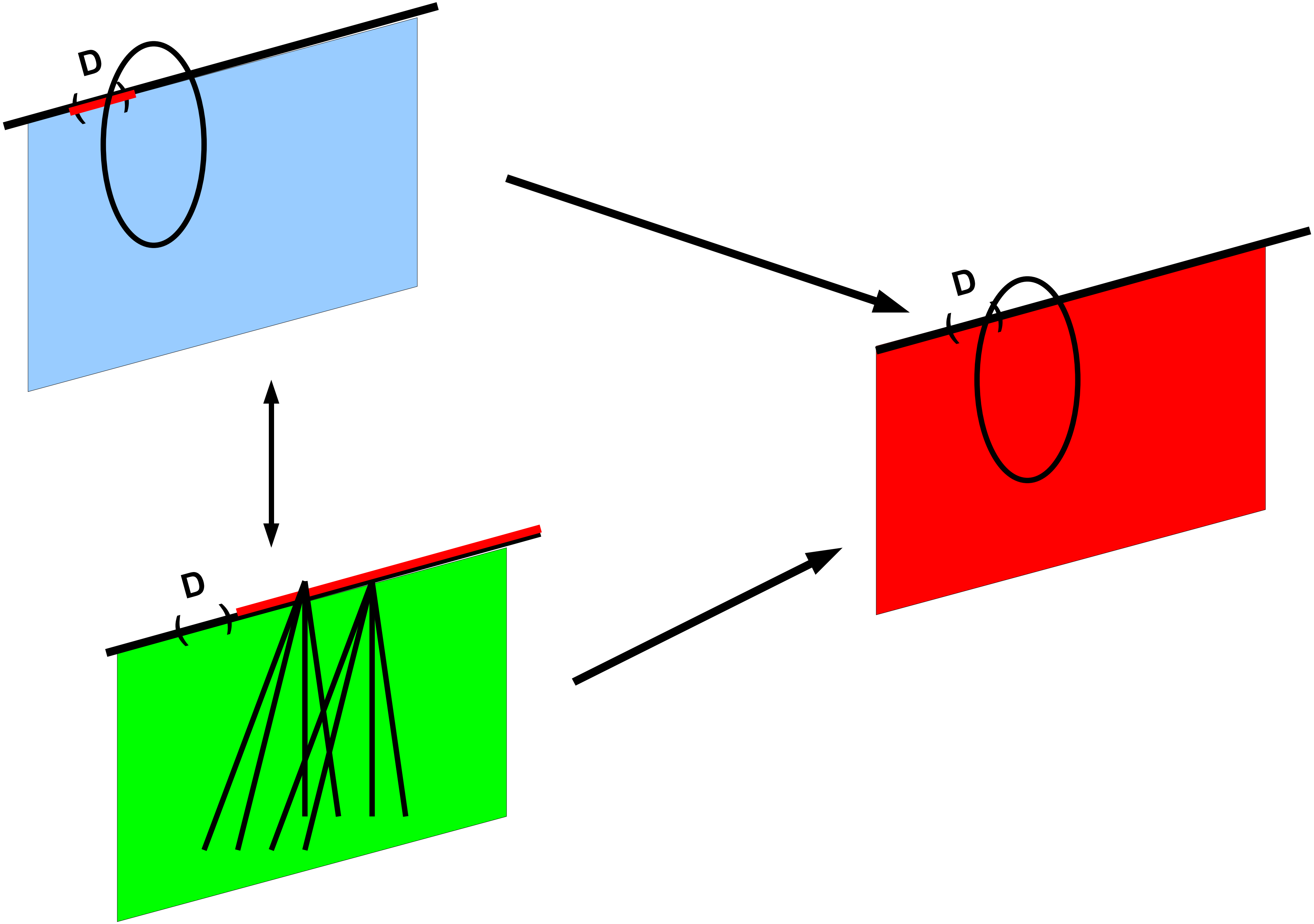
*modulo any points that may lie in the derivation set, since such points are not points of the Hall plane. In this situation we require only that the inherited arc outside of the derivation set in the Hall plane can be completed to an oval of the Hall plane.

Hall Planes of Even Order

In Hall planes of even order we extend the definition of inherited oval to that of **inherited hyperoval** with the same proviso for points in the derivation set.

There are several known results concerning inherited hyperovals in Hall planes of even order that we can extend and simplify.

O'Keefe-Pascasio-Penttila Type



O'Keefe-Pascasio-Penttila

Theorem 1: Let $\mathscr{H}$ be a translation hyperoval with tangent point T and nucleus N on ℓ_∞, one and only one of which is in D. $\mathscr{H}$ is an inherited translation hyperoval in the Hall plane $H(q^2)$.

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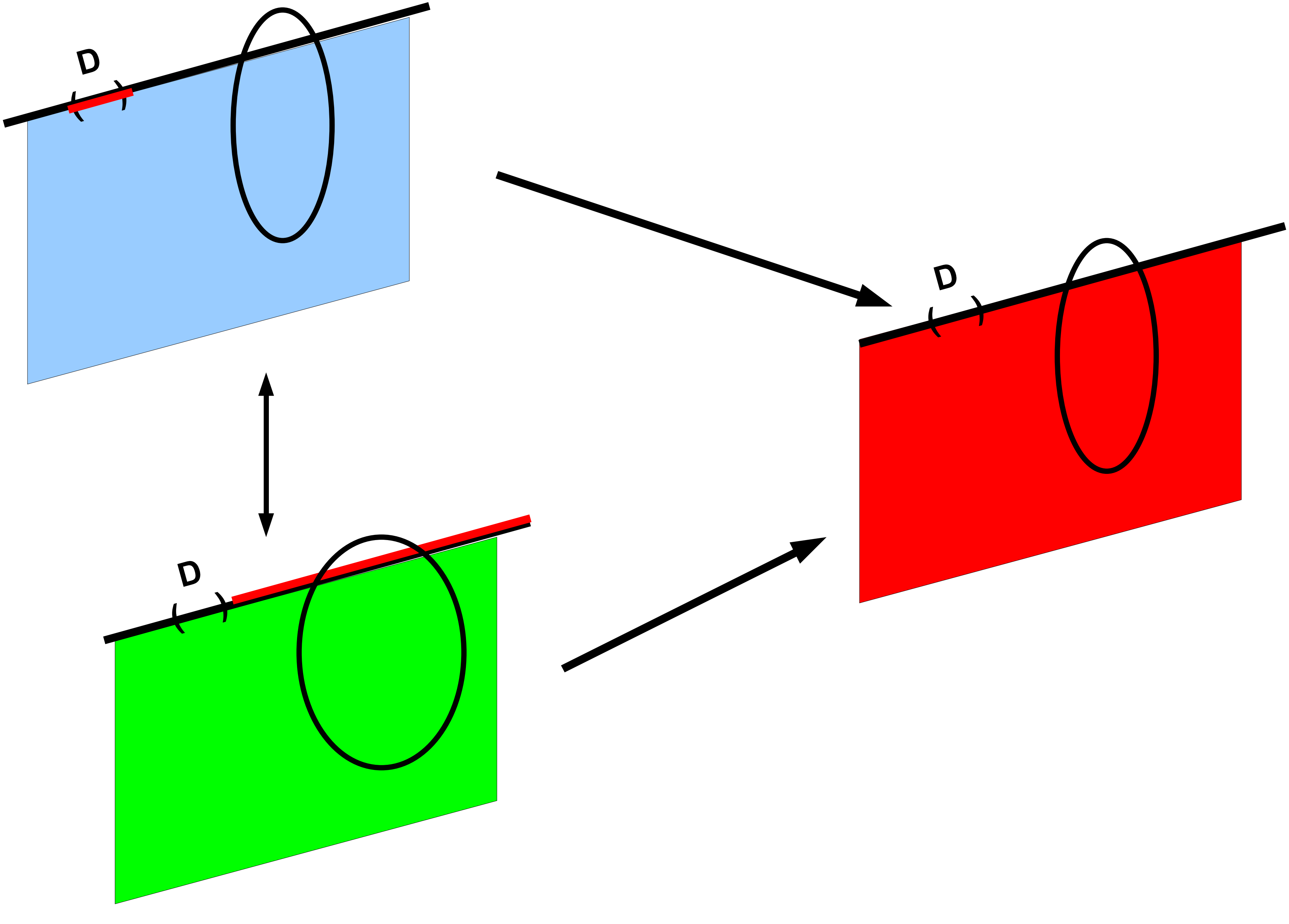
O'Keefe, Pascasio & Penttila proved this result in 1992 for the case of hyperconics (regular hyperovals).

Crismale gave some examples of translation hyperovals of this type in 1981.

Glynn-Steinke Type



Glynn-Steinke

Each derivation set has a unique involutory mapping which fixes each point of the derivation set. Points on ℓ_∞ outside of the set are called ***conjugate*** if they are images of each other under this mapping.

Theorem 2: Let $\mathscr{H}$ be a translation hyperoval with T and N conjugate points with respect to D. $\mathscr{H}$ is an inherited hyperoval iff q is a square.

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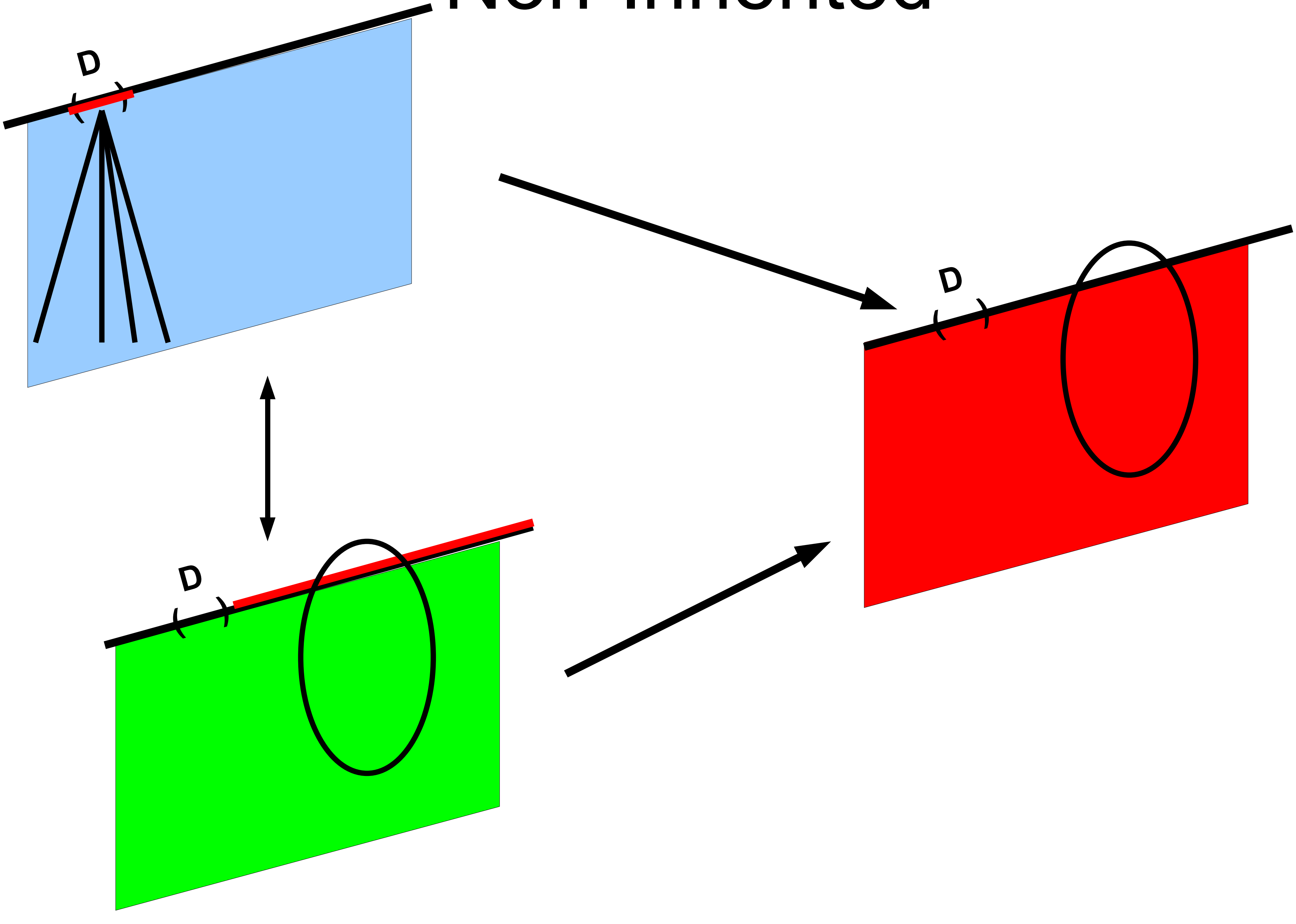
Glynn & Steinke proved this for conics in 1993.

Denniston (1971) gave a construction of translation hyperovals in André planes of even order. This gives a Glynn-Steinke type hyperoval in Hall(16).

Non-Inherited



Non-Inherited

Theorem 3: Let S be the set of affine points of A_1 satisfying the equation

$$\mathbf{a^r A x^{qr} + A y^r + b x^q + y = 0}$$

with $A \in GF(q^2)$, q a square, $A \neq 0$, $r = 2^i$ with $(i,2h) = 1$, a and b belonging to different proper cosets of D, and if $q > 4$, $a/b \in GF(q)$. Then, any line of A_1 which intersects S in more than two points has slope in D and in the Hall plane $H(q^2)$, S is a q^2-arc which extends to a translation hyperoval of $H(q^2)$.

Hall Planes of Odd Order

For odd q we have the following cases for inherited ovals:

a) ℓ_∞ is a secant line with points of intersection P and Q;

i) P and Q are both in D,

ii) one, but not both, of P and Q is in D,

iii) neither is in D.

b) ℓ_∞ is a tangent line;

c) ℓ_∞ is an exterior line.

Case a(i) was dealt with by O'Keefe and Pascasio (1996) who showed non-existence for $q > 3$ (but complete q^2-1 arcs can occur).

Korchmàros (1986) showed non-existence in case (b).

Case a(ii)

Theorem 4: There are no inherited conics having ℓ_∞ as a secant line with exactly one of its conic points in the derivation set.

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The proof involves examining (in A_2) the intersections of the curve $y = x^{q+1}$ with conics.

Case a(iii) – Neither point in D

Theorem 5: If *O* is a conic projectively equivalent to one with equation $xy = k$ by a projectivity which preserves D, then *O* is inherited if and only if k is a non-square in $GF(q^2)$.

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The inherited ovals of this theorem are analogous to the Glynn-Steinke type inherited hyperovals in the even case, since hyperbolas with $P = (1,c,0)$ and $Q = (1,c^{-q},0)$ (conjugate wrt D) are of this type.

Case (c) : Elliptic Case

Conjecture: There are no inherited ellipses in Hall planes of odd order.

Theorem 6: A conic with an equation of the form $Ax^2 + By^2 + 1 = 0$ where AB is a nonsquare is not an inherited conic in the Hall plane if there is an exterior point of the conic in D.

Non-inherited Ovals

In the odd order situation, there are no known examples of non-inherited ovals in Hall(q^2). However, complete searches have been made only in

Hall(9) [Denniston (1971), Ninzette (1971), Killgrove, Kiel and Koch (1979)]

and

Hall(25) [Ryan Pedersen (2008)].