

# Ovals and Hyperovals in Hall Planes

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# Hall Planes by Derivation

A Hall plane  $H(q^2)$  is constructed from  $PG(2, q^2)$  by a process called **derivation**.

This requires a **derivation set**  $D$  – a set of  $q+1$  points on  $\ell_\infty$  such that the set of all subplanes of order  $q$  (**Baer subplanes**) that contain  $D$  has the property that for any two points of the affine plane that are on a line with slope in  $D$  are contained in one of these subplanes.

It can be shown that  $D = \{\infty\} \cup \{(m) : m \in GF(q)\}$  is a derivation set. All derivation sets are equivalent under the group of the plane which preserves  $\ell_\infty$ .

# Hall Planes by Derivation

To obtain the Hall plane with respect to a derivation set  $D$ , we remove all the lines of  $\text{PG}(2, q^2)$  which intersect  $D$  (this includes  $\ell_\infty$ ) and declare all the Baer subplanes which belong to  $D$  to be lines. This produces an affine plane which we extend to a projective plane in the usual way. The resulting Hall plane is non-Desarguesian iff  $q > 2$ .

*Note:* Collineations of  $\text{PG}(2, q^2)$  which preserve  $D$  will also be collineations of  $H(q^2)$ .

# More on Derivation Sets

We will not use the “standard” derivation set, rather the derivation set

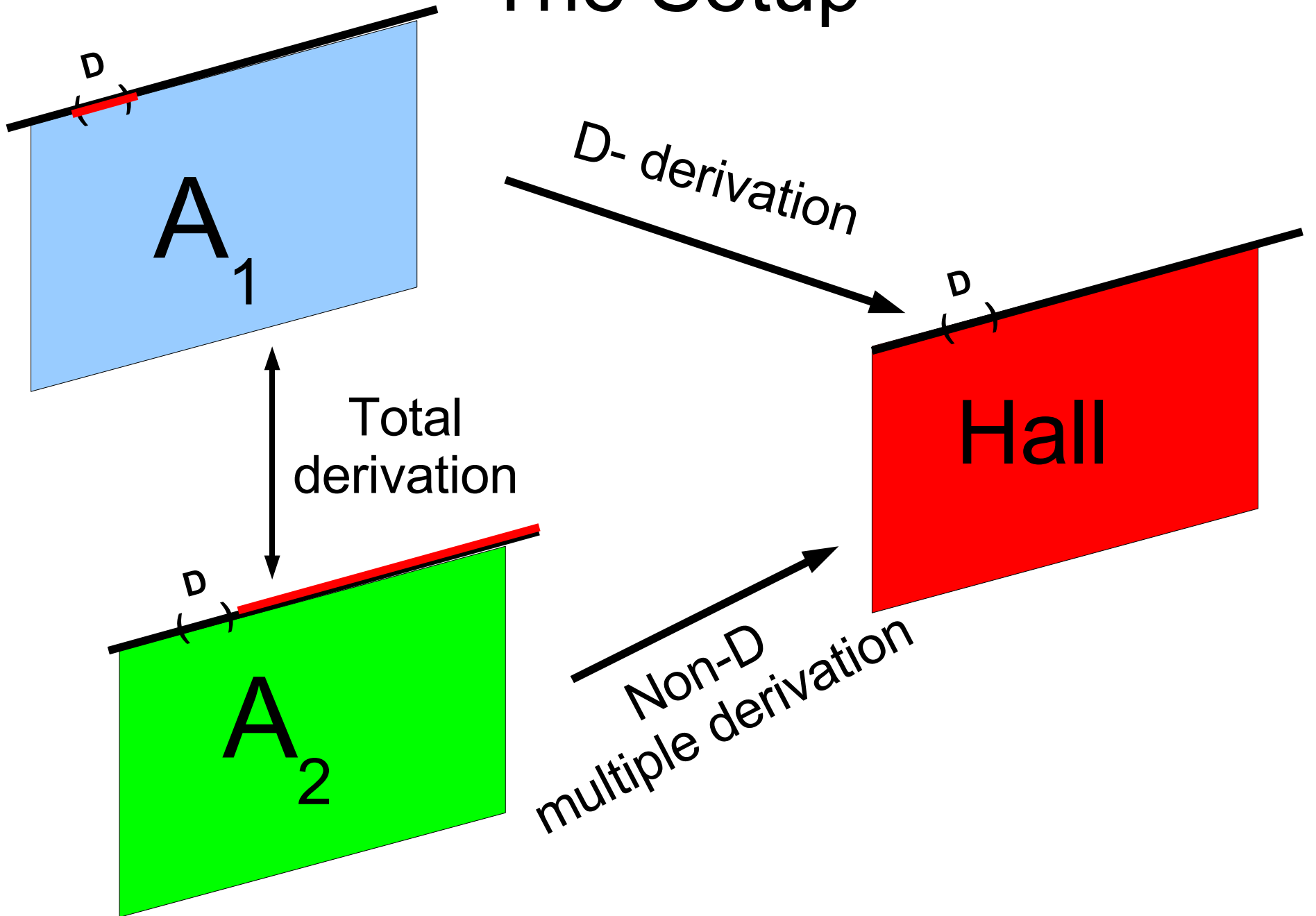
$$D = \{(n) : n^{q+1} = 1\}.$$

All the proper cosets of this multiplicative group are also derivation sets (and the elements of  $GF(q) \setminus \{0\}$  can be taken as representatives if  $q$  is even).

One can independently derive with respect any selection of these disjoint derivation sets. This is called multiple derivation and produces (generalized) André planes.

Multiple derivation using all these derivation sets gives a Desarguesian plane, and all except one gives a Hall plane.

# The Setup



# Some Details

With our choice of  $D$  we have:

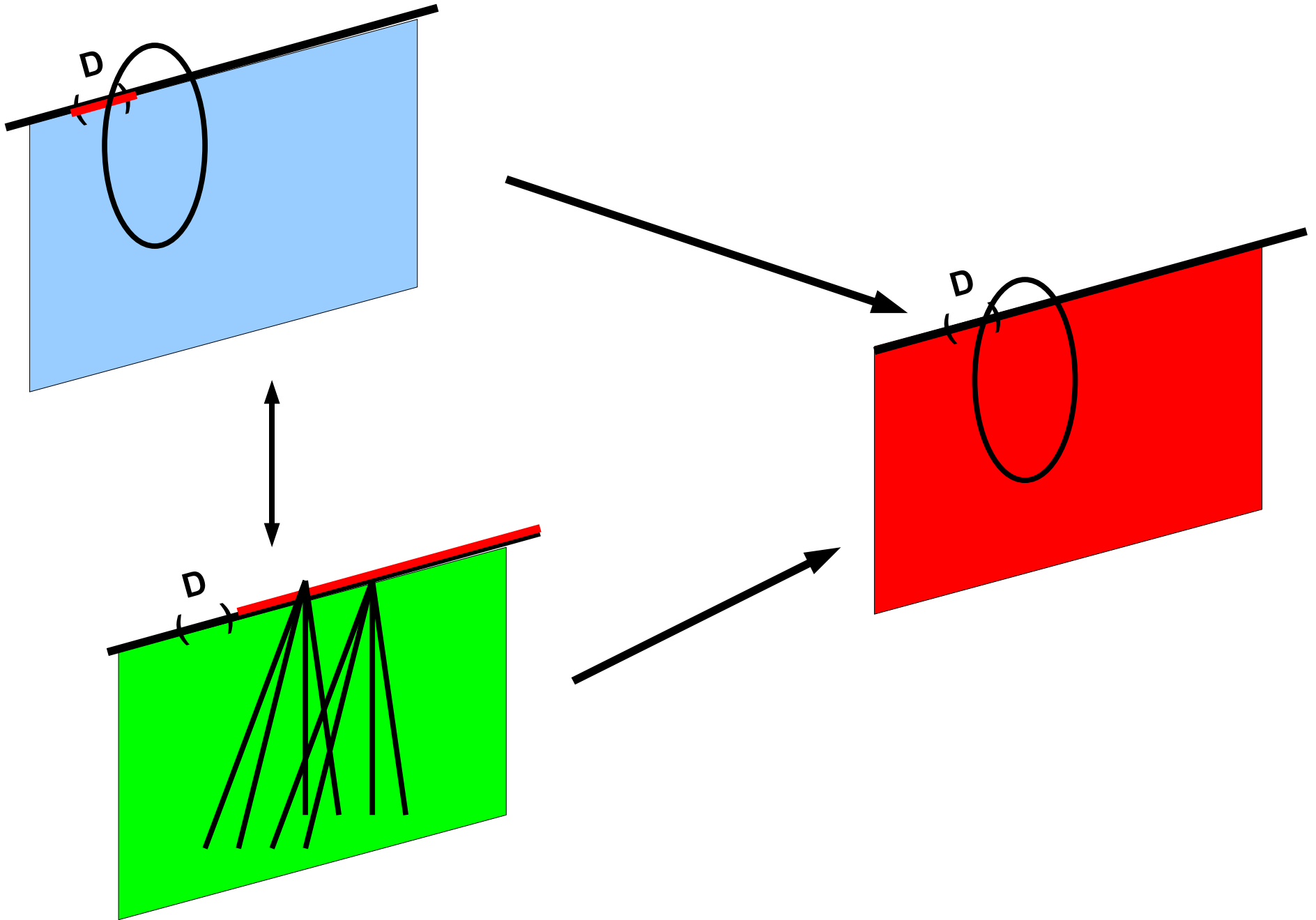
- 1) Points of Baer subplane belonging to  $D$  satisfy the equation  $\mathbf{y} = \mathbf{A}\mathbf{x}^q + \mathbf{B}$  with  $A \in D$ ,  $B \in GF(q^2)$ .
- 2) Total derivation between  $A_1$  and  $A_2$  is given by the involutory map  $(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{x}^q, \mathbf{y})$ .
- 3) The points  $(0,0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$  in the same Baer subplane  $y = Ax^q$  are collinear iff  $\mathbf{x}_1^{q-1} = \mathbf{x}_2^{q-1}$ .

# Inherited Ovals

An (hyper)oval in  $PG(2, q^2)$  whose point set is an (hyper)oval in the corresponding Hall plane is called an *inherited (hyper)oval*.

There are several results concerning inherited ovals. We shall be concerned only with even order planes. All known results concern ovals with  $\ell_\infty$  as a tangent line, which means that the nucleus of the oval is also on  $\ell_\infty$ . Thus, for the remainder of the talk we shall assume  $q = 2^h$ .

# O'Keefe-Pascasio-Penttila



# O'Keefe-Pascasio-Penttila

**Theorem 1:** Let  $\mathcal{H}$  be a translation hyperoval with tangent point  $T$  and nucleus  $N$  on  $\ell_\infty$ , one and only one of which is in  $D$ .  $\mathcal{H}$  is an inherited translation hyperoval in the Hall plane  $H(q^2)$ .

*Outline of Proof:* There are two cases to consider, but they are similar so we will only deal with the case that  $N$  is in  $D$ .

By a collineation which preserves  $D$  we can arrange to have  $T = (\infty)$  and  $N = (1)$  and have  $\mathcal{H}$  contain the origin. The affine points of  $\mathcal{H}$  then satisfy the equation

$$Ax^r + x + y = 0, \quad A \in GF(q^2) \setminus \{0\}, \quad r = 2^i \text{ with } (i, 2h) = 1.$$

# O'Keefe-Pascasio-Penttila

The corresponding points in  $A_2$  satisfy:

$$Ax^{qr} + x^q + y = 0.$$

The x coordinates of the points on this curve and the line  $y = mx$  are solutions of:

$$Ax^{qr} + x^q + mx = 0.$$

If  $m \in D$  then

$$x^q + mx = m(x^q + mx)^q.$$

A substitution leads to,

$$\mathbf{x^{q-1} = (mA^{q-1})^{1/r}} \text{ a constant, for } x \neq 0.$$

If 3 points of this curve in  $A_2$  lie on a line with slope  $m \in D$ , then there are 2 points collinear with the origin on  $y = mx$ . The corresponding points in  $A_1$  lie in a Baer subplane and are collinear points on the hyperoval  $\mathcal{H} \rightarrow \leftarrow$

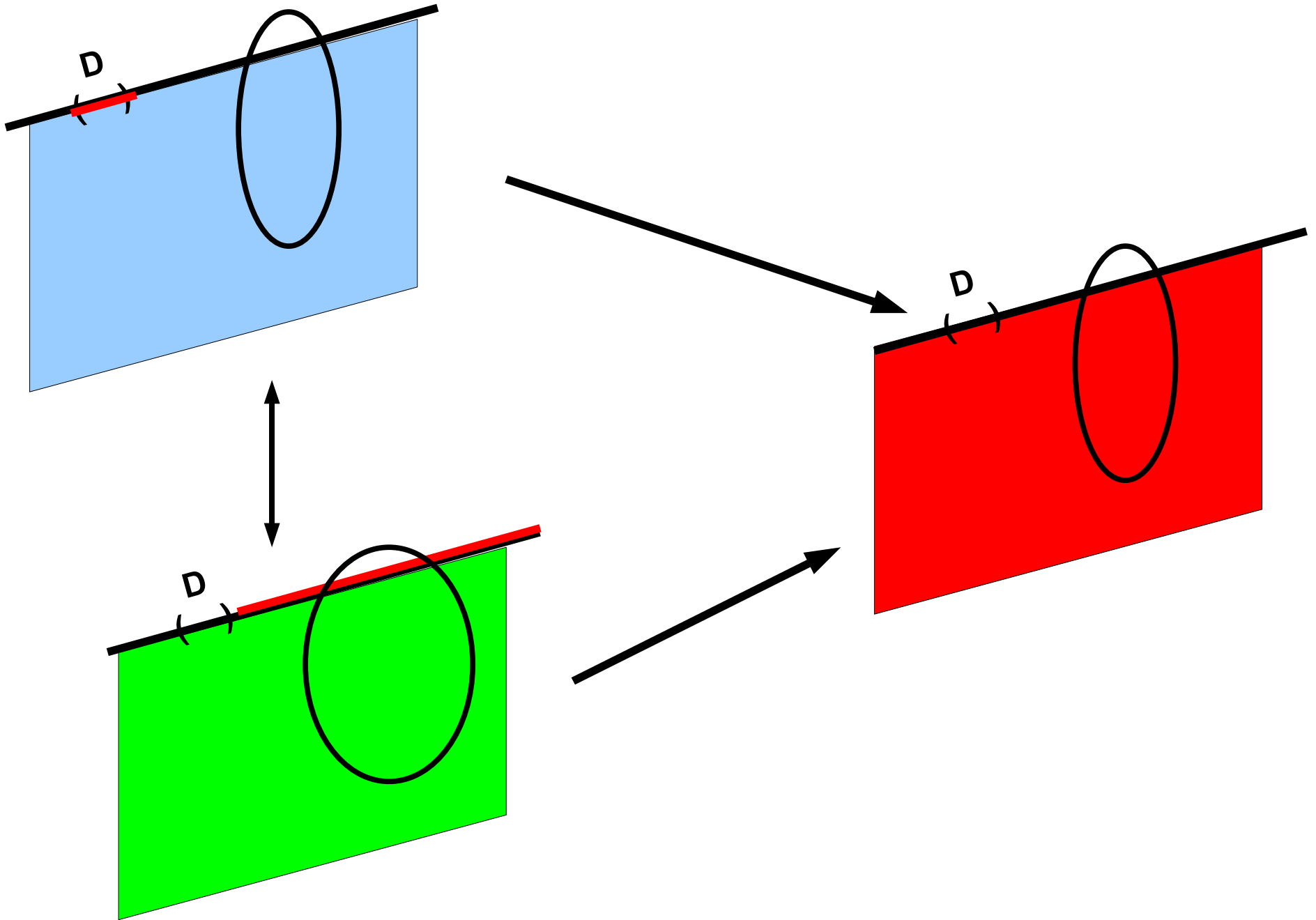
Thus, in  $A_2$  no more than 2 points of the curve lie on a line with slope in  $D$ , so  $\mathcal{H}$  is an inherited oval.  $\square$

# O'Keefe-Pascasio-Penttila

O'Keefe, Pascasio & Penttila proved this result in 1992 for the case of conics ( $r = 2$ ).

Crismale gave some examples of translation hyperovals of this type in 1981.

# Glynn-Steinke



# Glynn-Steinke

Each derivation set has a unique involutory mapping which fixes each point of the derivation set. Points on  $\ell_\infty$  outside of the set are called **conjugate** if they are images of each other under this mapping.

**Theorem 2:** Let  $\mathcal{H}$  be a translation hyperoval with T and N conjugate points with respect to D.  $\mathcal{H}$  is an inherited hyperoval iff  $q$  is a square.

*Proof:* We can arrange to have  $T = (\infty)$  and  $N = (0)$  and  $\mathcal{H}$  containing the origin.  $\mathcal{H}$  then has an equation of the form:

$$Ax^r + y = 0, \quad A \in \text{GF}(q^2) \setminus \{0\}, \quad r = 2^i \text{ with } (i, 2h) = 1.$$

The corresponding set in  $A_2$  satisfies:

$$Ax^{qr} + y = 0.$$

This is a translation oval in  $A_2$  iff  $q$  is a square.  $\square$

# Glynn-Steinke

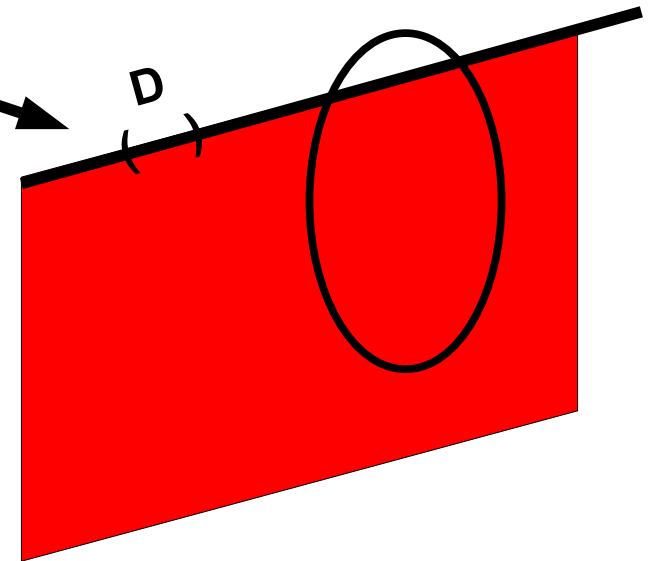
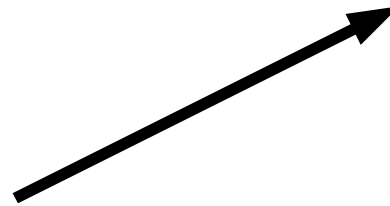
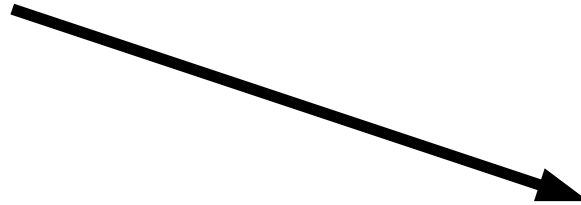
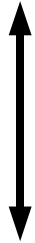
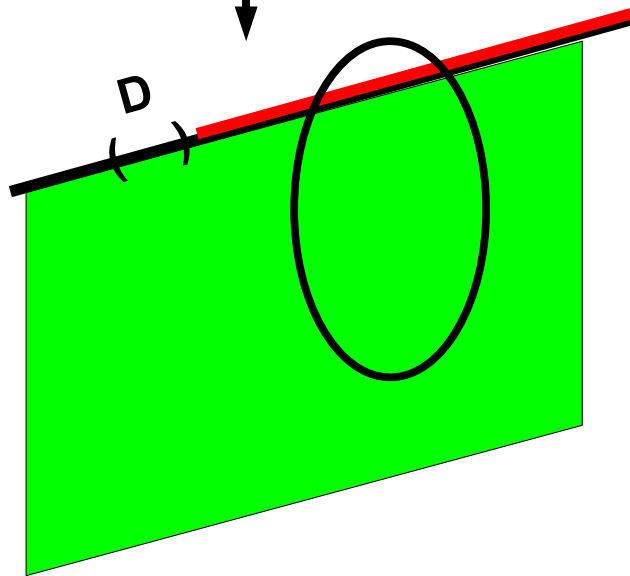
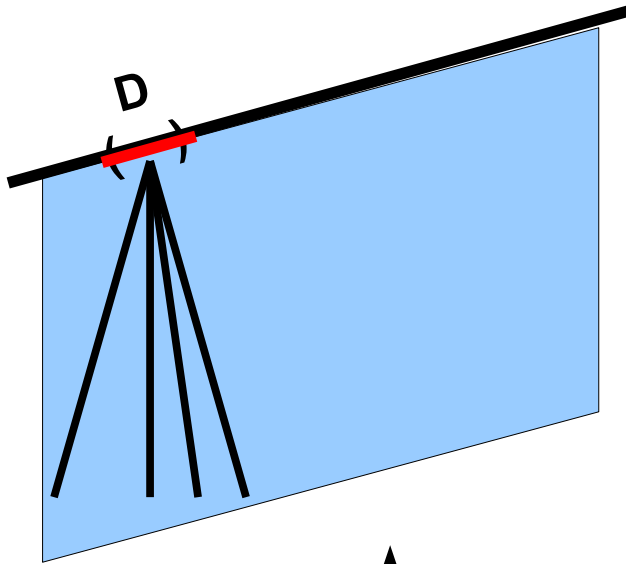
Glynn & Steinke proved this for conics ( $r = 2$ ) in 1993 in a 7 page paper.

Denniston (1971) gave a construction of translation hyperovals in André planes of even order. This gives a Glynn-Steinke type hyperoval in Hall(16).

Our proof has an immediate corollary:

**Corollary:** Let  $\mathcal{H}$  be a translation hyperoval with  $T$  and  $N$  conjugate points with respect to  $D$ .  $\mathcal{H}$  is an inherited hyperoval in any André plane obtained by multiple derivation with respect to the cosets of  $D$  iff  $q$  is a square.

# $A_2$ -Inherited



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**Theorem 3:** Let  $S$  be the set of affine points of  $A_1$  satisfying the equation

$$\mathbf{a^rAx^{qr} + Ay^r + bx^q + y = 0}$$

with  $A \in GF(q^2)$ ,  $q$  a square,  $A \neq 0$ ,  $r = 2^i$  with  $(i, 2h) = 1$ ,  $a$  and  $b$  belonging to different proper cosets of  $D$ , and if  $q > 4$ ,  $a/b \in GF(q)$ . Then, any line of  $A_1$  which intersects  $S$  in more than two points has slope in  $D$  and in the Hall plane  $H(q^2)$ ,  $S$  is a  $q^2$ -arc which extends to a translation hyperoval of  $H(q^2)$ .

*Proof:* In  $A_2$  the corresponding point set satisfies

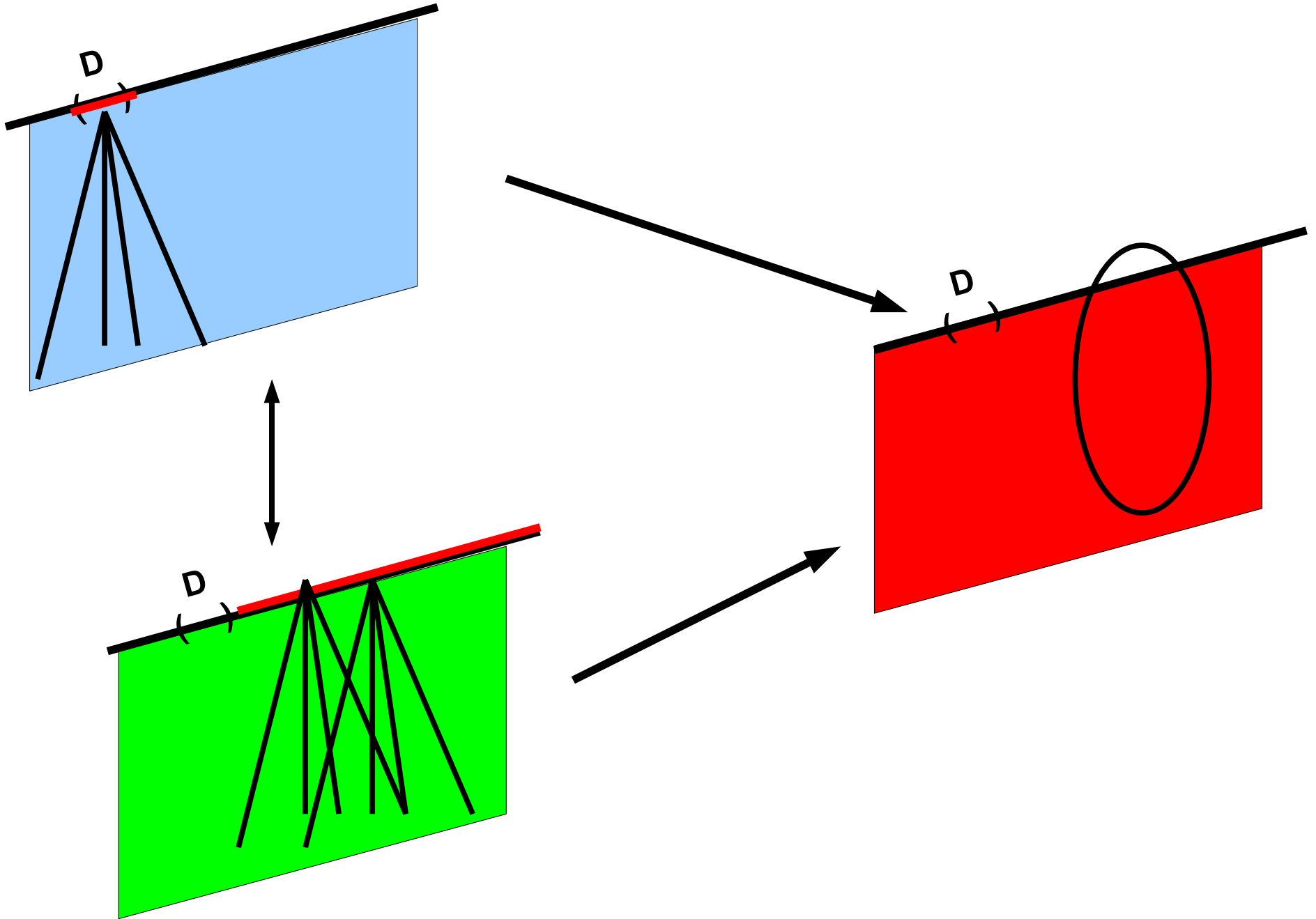
$$\mathbf{a^rAx^r + Ay^r + bx + y = 0}$$

and lies on a translation hyperoval with  $T = (1, a, 0)$  and  $N = (1, b, 0)$ , conjugate with respect to one of the proper cosets of  $D$  if  $q > 4$ .

# $A_2$ -Inherited

If  $q > 4$ , perform multiple derivation on  $A_2$  with respect to all the cosets other than  $D$  and the ones containing  $T$  and  $N$  (if  $q = 4$ , skip this step). By the corollary, this translation hyperoval is inherited in the resulting André plane, having the same  $T$  and  $N$ . Now, use the coset containing  $N$  to derive this André plane. Since only one of  $N$  or  $T$  is in the derivation set, the translation hyperoval is again inherited by Theorem 1 (which is valid in André planes).  $N$  may change to  $N'$ , but  $N'$  is in the same coset as  $N$ . Now, derive the new André plane with respect to the coset containing  $T$ . Again by Theorem 1, this translation hyperoval is inherited in the new plane. But, this last plane is obtained from  $A_2$  by non- $D$  multiple derivation, it is the Hall plane.  $\square$

# Non-inherited



# Non-inherited

We will restrict ourselves to  $PG(2,16)$ , but the construction technique (alas, not the proof) most likely generalizes to higher orders. Throughout this section  $q = 4$ .

We give the construction:

Let  $\eta$  be an element of norm 1 other than 1. ( $\eta^{q+1} = 1$ )

Consider two conics, having affine equations:

$$\mathcal{C}_1 : Ax^2 + (\eta+1)x + y = 0, \text{ and}$$

$$\mathcal{C}_2 : Ax^2 + (\eta+1)x + y + (\eta+1)/A = 0,$$

where  $A \neq 0$ .

**Note:** These have the same point of tangency and nucleus.

# Non-inherited Construction

The idea is to pick  $q^2/2$  affine points from each conic so that the resulting set has the appropriate structure.

Let  $T_0$  be the set of elements of absolute trace 0 in  $GF(q^2)$ , and define

$$G_1 = \frac{\eta^2 + \eta + 1}{A} T_0 \quad \text{and} \quad G_2 = GF(q^2) \setminus G_1$$

Our set  $\mathcal{S}$  is constructed by taking the points of  $\mathcal{C}_1$  whose x-coordinates lie in  $G_1$  and the points of  $\mathcal{C}_2$  whose x-coordinates lie in  $G_2$ .

# Properties of $\mathcal{S}$

1.  $\mathcal{S}$  is a translation set.

$\mathcal{S}$  is invariant under the translations  $(x,y) \rightarrow (x+a,y+b)$  for every  $(a,b) \in \mathcal{S}$ .

2. There are  $q^2/4$  lines with slope 1 which meet  $\mathcal{S}$  in 4 points and all other lines meet  $\mathcal{S}$  in no more than 2 points.

3. The vertical lines and lines with slope  $\eta + 1/\eta$  are all tangent lines to  $\mathcal{S}$ .

4. The lines of  $A_2$  with slope in  $D$  meet the image of  $\mathcal{S}$  in no more than 2 points.

5. In Hall(16)  $\mathcal{S}$  extends to a translation hyperoval.

# Hall(16)

A complete search for hyperovals in Hall(16) was made in 1991, but the table reporting the results was incorrect.

Hyperovals of Hall(16) containing the origin

Type	Class	Number	Group
H1 (10)	-----	None	-----
H2 (60)	I	15	$T \rtimes Z_4$
	II	15	$T \rtimes Z_4$
H3 (60)	I	60	T
H4 (6)	I	30	$T \rtimes (Z_5 \rtimes Z_4)$

# Hall(16)

We can now identify the hyperovals in this plane.

H1 – two infinite points in the derivation set.

H2 – O'Keefe-Pascasio-Penttila  
the two classes correspond to tangent point being in  
or out of the derivation set.

H3 – Non-inherited and  $A_2$  inherited  
these are projectively equivalent in Hall(16)

H4 – Glynn-Steinke (6 pairs of conjugate infinite points)