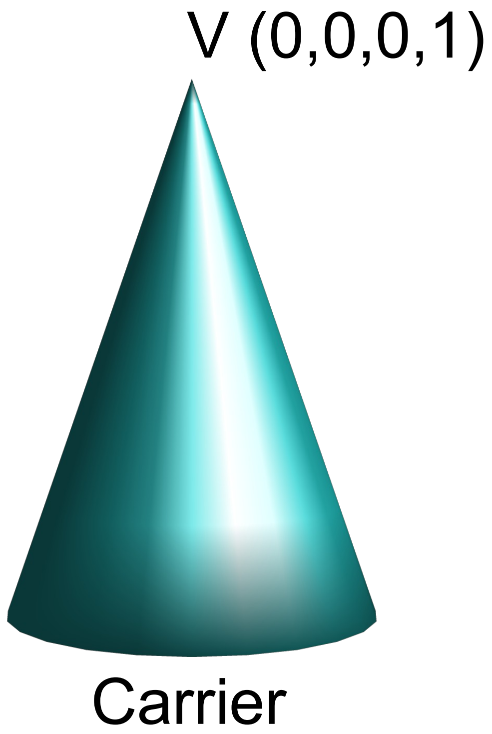
Bi-Linear Flocks and Translation Planes

Bill Cherowitzo
Univ. Colorado Denver
June 20, 2009
Rocky Mountain Discrete Mathematics Day 2009

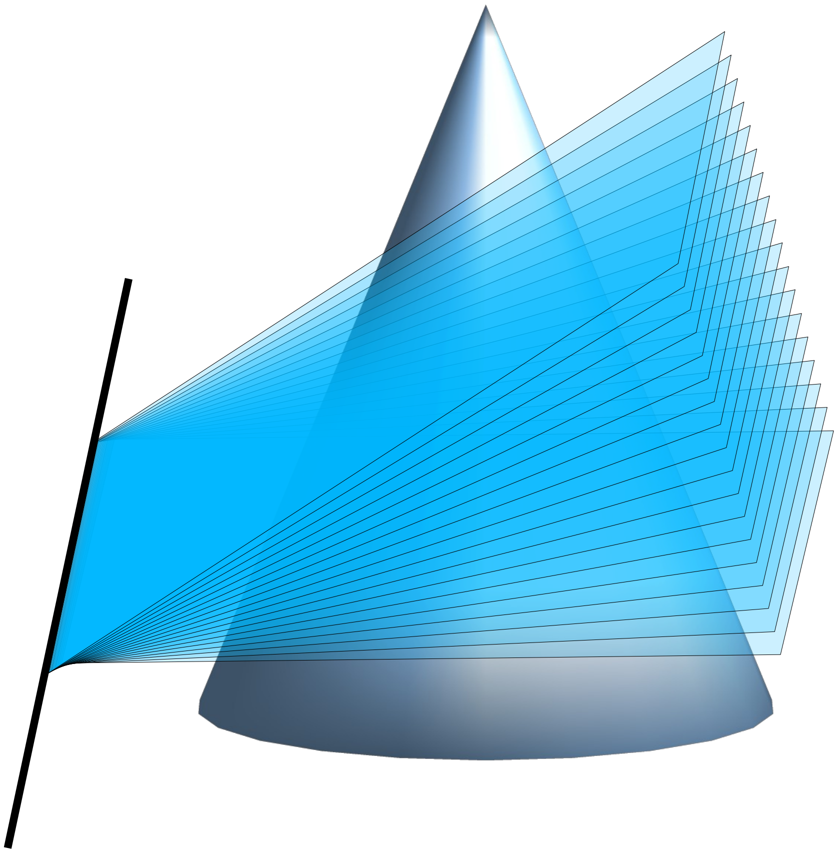
Flocks of Arbitrary Cones



A ***flock*** of a cone in PG(3,q) is a set of q planes which partition the points of the cone (except for the vertex).

The ***carrier*** (or base) of the cone may be any set of points in the plane $x_3 = 0$, which can always be taken as one of the planes in the flock.

Linear Flocks



The Flock Condition

A set of q planes (indexed by elements of GF(q)) not passing through V(0,0,0,1) can be described by three functions f,g,h: GF(q) → GF(q) defined by the equations of the planes;

$$\pi_t : f(t)x_0 + g(t)x_1 + h(t)x_2 - x_3 = 0.$$

Such a set of q planes is a flock of the cone with carrier $\mathcal{C}$ if and only if

$$t \rightarrow f(t)a + g(t)b + h(t)c$$

is a permutation of GF(q) for each $(a,b,c,0) \in \mathcal{C}$.

Denote a flock by $\mathcal{F} = \mathcal{F}(f,g,h)$.

Translation Planes and Spread Sets

A ***translation plane*** is a projective plane whose translation group acts regularly on its affine points.

A (matrix) ***spread set*** on $GF(q)^2$ is a set of q^2 2x2 matrices, including the zero matrix, such that the difference of any two distinct matrices is invertible.

By a standard construction, every spread set gives rise to a translation plane. Any translation plane will have many spread set representations.

Flokki and Spread Sets

The cone over carrier $xz^{\alpha} = y^{\alpha+1}$ (with $x \to x^{\alpha}$ an automorphism) has a flock $\mathcal{F} = \mathcal{F}(-f^{\alpha}, g^{\alpha}, t)$ iff

$$-x^{\alpha+1}f(t)^{\alpha} + xg(t)^{\alpha} + t \text{ is a permutation } \forall\, x.$$

Using $\Delta f = f(t_1)-f(t_2)$, $\Delta g = g(t_1)-g(t_2)$ and $\Delta t = t_1-t_2$ we have that

$$-x^{\alpha}(\Delta f)^{\alpha} + (\Delta g)^{\alpha} + (\Delta t)/x \neq 0 \quad \text{if } t_1 \neq t_2.$$

With $u^{\alpha} = (\Delta t)/x$, this is equivalent to

$$u^{\alpha}(u + \Delta g) - \Delta t \Delta f \neq 0 \text{ if } t_1 \neq t_2.$$

$$\begin{pmatrix} u+g(t) & f(t) \\ t & u^{\alpha} \end{pmatrix}$$ is a spread set.

A special case

Consider the special case of flokki in PG(3,q^2) where we take α = q. A flock of the form $\mathcal{F}(-f^q,0,t)$ gives rise to the spread set:

$$\begin{pmatrix} u & f(t) \\ t & u^q \end{pmatrix}, \quad u,t \in GF(q^2).$$

There is a technique, called “algebraic lifting” which will take ***any*** spread over GF(q) and produce one of the above form.

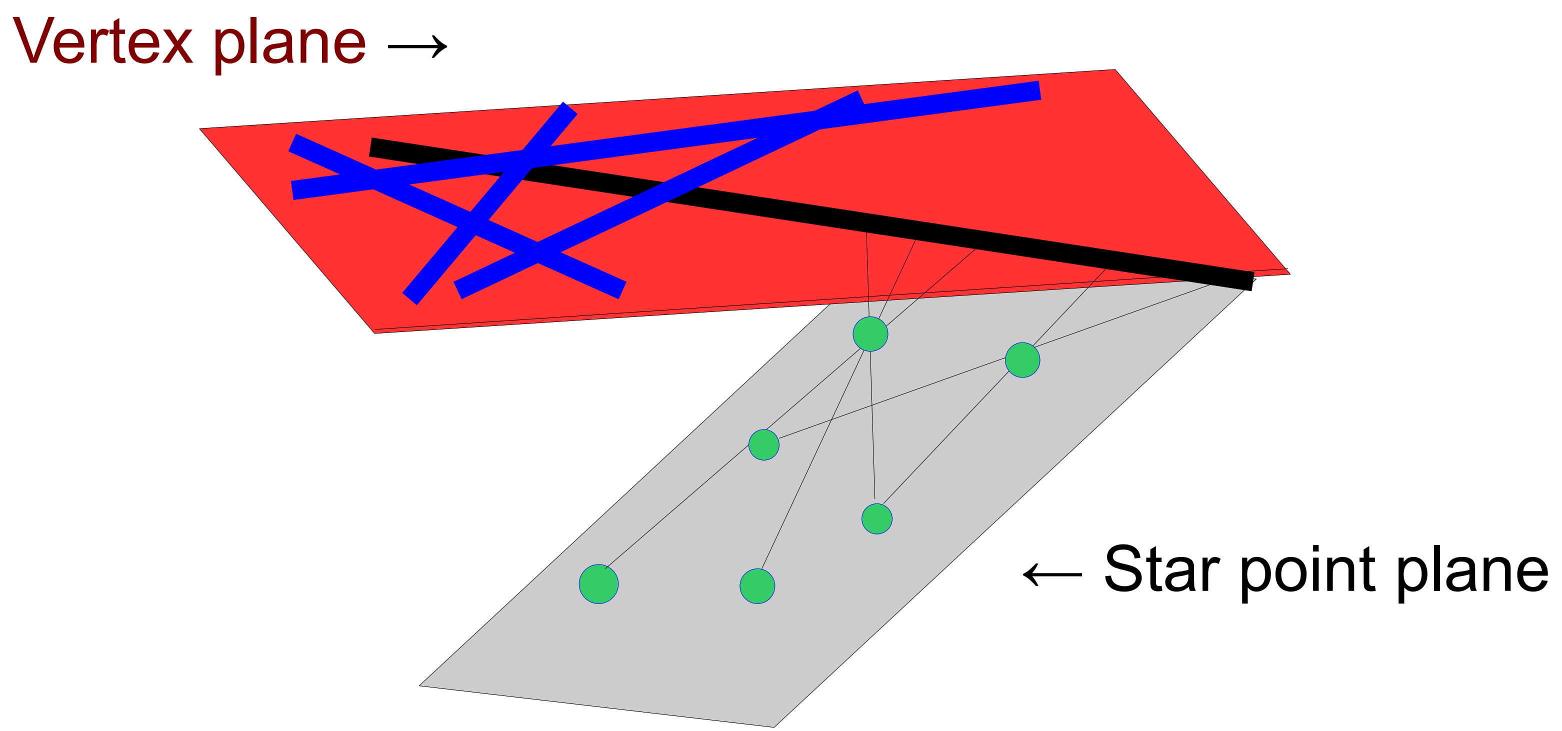
Star Flocks

A flock in which all the planes meet at a unique point is called a (proper) ***star flock***. The intersection point is called the ***star point***.

A flock $\mathcal{F}$(f,g,h) in which one of the functions is the zero function is a star flock.

Star flocks are most easily studied by considering the space dual, where points and planes are interchanged and lines get mapped to lines. In this setting, a flock consists of q points in the plane corresponding to the star point, none of which is in the plane corresponding to the vertex of the cone.

Dual Setting for Star Flocks



Line joining two flock points does not meet any line which is the image of a generator of the cone.

Rédei Blocking Sets

A ***blocking set*** is a set of points in a plane which every line of the plane intersects.

A ***Rédei blocking set*** is a blocking set consisting of q+N points in a plane of order q containing a largest collinear set of N points (called a ***Rédei line***).

Let a Rédei line be the line at infinity of the plane. The set $\mathcal{V}$ of the q affine points of this blocking set has the property that the join of any of its points meets the line at infinity in a blocking set point.

The “Big” Theorem

Theorem:(Ball, 2002) Let f be a function from GF(q) to GF(q), $q = p^h$ for some prime p, and let N be the number of directions determined by f. Let $s = p^e$ be maximal such that any line with a direction determined by f that is incident with a point of $\mathcal{U}$ is incident with a multiple of s points of $\mathcal{U}$. One of the following holds:

(i) $s = 1$ and $(q+3)/2 \le N \le q+1$;
(ii) GF(s) a subfield of GF(q) & $q/s + 1 \le N \le (q-1)/(s-1)$;
(iii) $s = q$ and $N = 1$.

Moreover, if $s > 2$ then f is an s-linearized polynomial.

Putting it all together

A star flock $\mathcal{F}(-f^q,0,t)$ of the cone over the carrier $xz^q = y^{q+1}$ in $PG(3,q^2)$ when viewed in the dual setting form the affine part of a Rédei blocking set having $N \le q^2 - q$ directions (slopes).

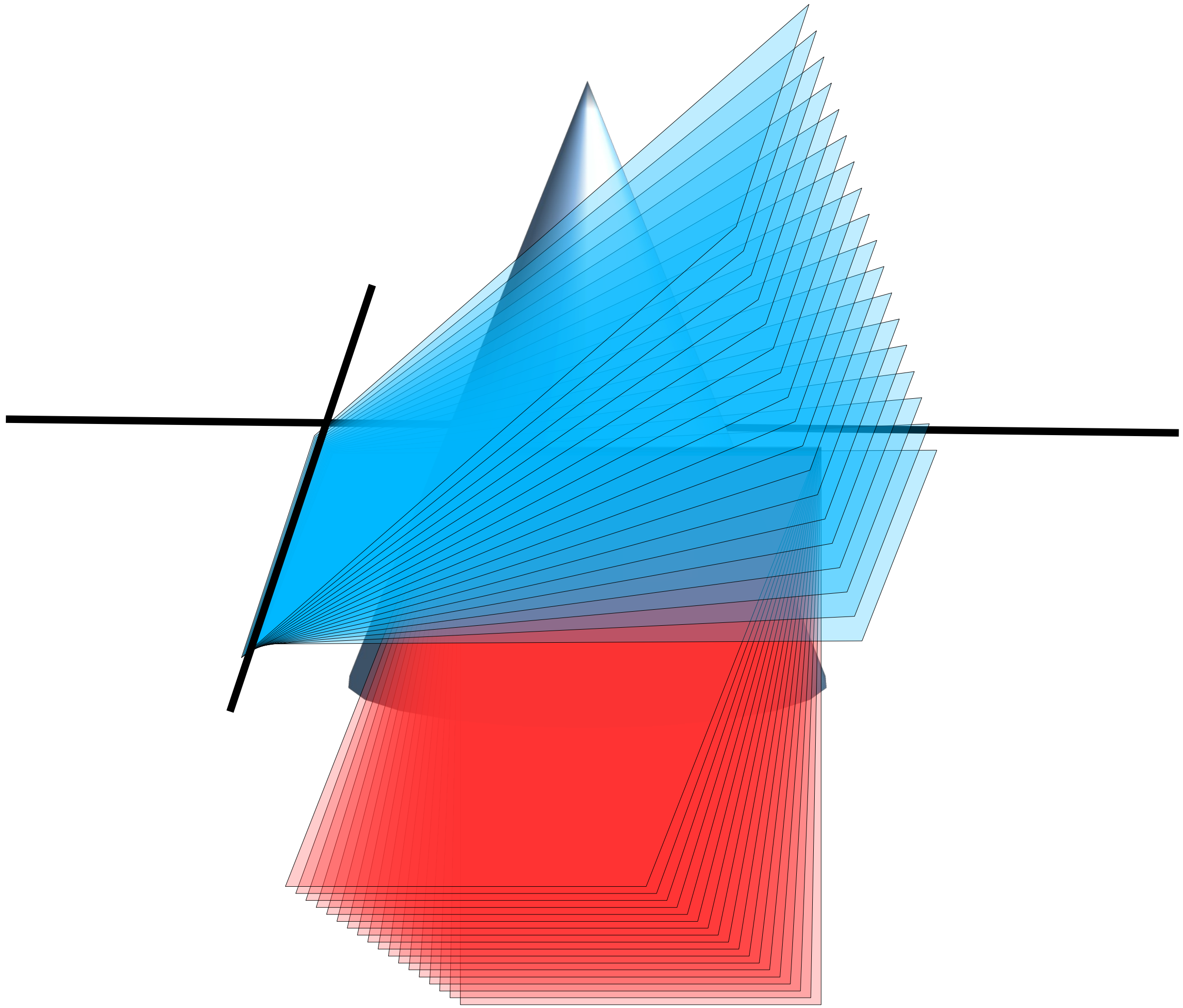
Note: The relationship between star flocks and Rédei blocking sets is general, it is only the bound on N which follows from this particular choice of cone.

Small Cases

q = 2: We would have $N \leq 2$, but by the "Big" Theorem, this implies $N = 1$. So, the star flock is a linear flock and the translation plane is the Desarguesian plane.

q = 3: We have $N \leq 6$ which implies $N = 1$, 4 or 6. As above $N = 1$ gives the linear flock and the Desarguesian plane. $N = 4$ gives an affine subplane of order 3. Since the function involved is 3-linearized, the translation plane is a semifield plane, but the only semifield plane of order 9 is the Desarguesian plane. This leaves the case $N = 6$ $(= q^2+3/2)$.

Bi-Linear Flocks



Bi-Linear Flocks

In PG(2,q), q odd, consider the q affine points (x,f(x)) where:

$$f(x) = x^{\frac{q^2+1}{2}} = \begin{cases} x & \text{if } \mathbf{x} = \text{☑} \\ -x & \text{if } \mathbf{x} = \text{☒} \end{cases}$$

These points lie on two lines and the number of slopes determined by any two points of the set is q+3/2. Together with the infinite points corresponding to these slopes, we have a Rédei blocking set with N = q+3/2 known as a ***projective triangle***.

... and Their Translation Planes

The spread sets obtained from these bi-linear flocks have:

$$f(t) = \sqrt{\alpha}\,(t^q)^{\frac{q^2+1}{2}}$$

where α is any non-zero element of GF(q) which is not a square in GF(q).

The translation planes are lifted regular nearfield planes. In the order 9 case, this is a Hall plane (which happens to also be a nearfield plane).