Combinatorial Designs:
Latin Squares

An Old Problem

Arrange the 16 face cards of a deck of playing cards in a 4 x 4 array so that each denomination (Ace, King, Queen, Jack) and each suit (Clubs, Hearts, Diamonds, Spades) appears only once in each row and column.

♠A	♥K	♦Q	♣J
♦J	♣Q	♠K	♥A
♣K	♦A	♥J	♠Q
♥Q	♠J	♣A	♦K

An enumeration by type of the solutions to this problem was published in **1723**.

Latin Squares

If we seperate the denominations and the suits we obtain:

♠	♥	♦	♣
♦	♣	♠	♥
♣	♦	♥	♠
♥	♠	♣	♦

A	K	Q	J
J	Q	K	A
K	A	J	Q
Q	J	A	K

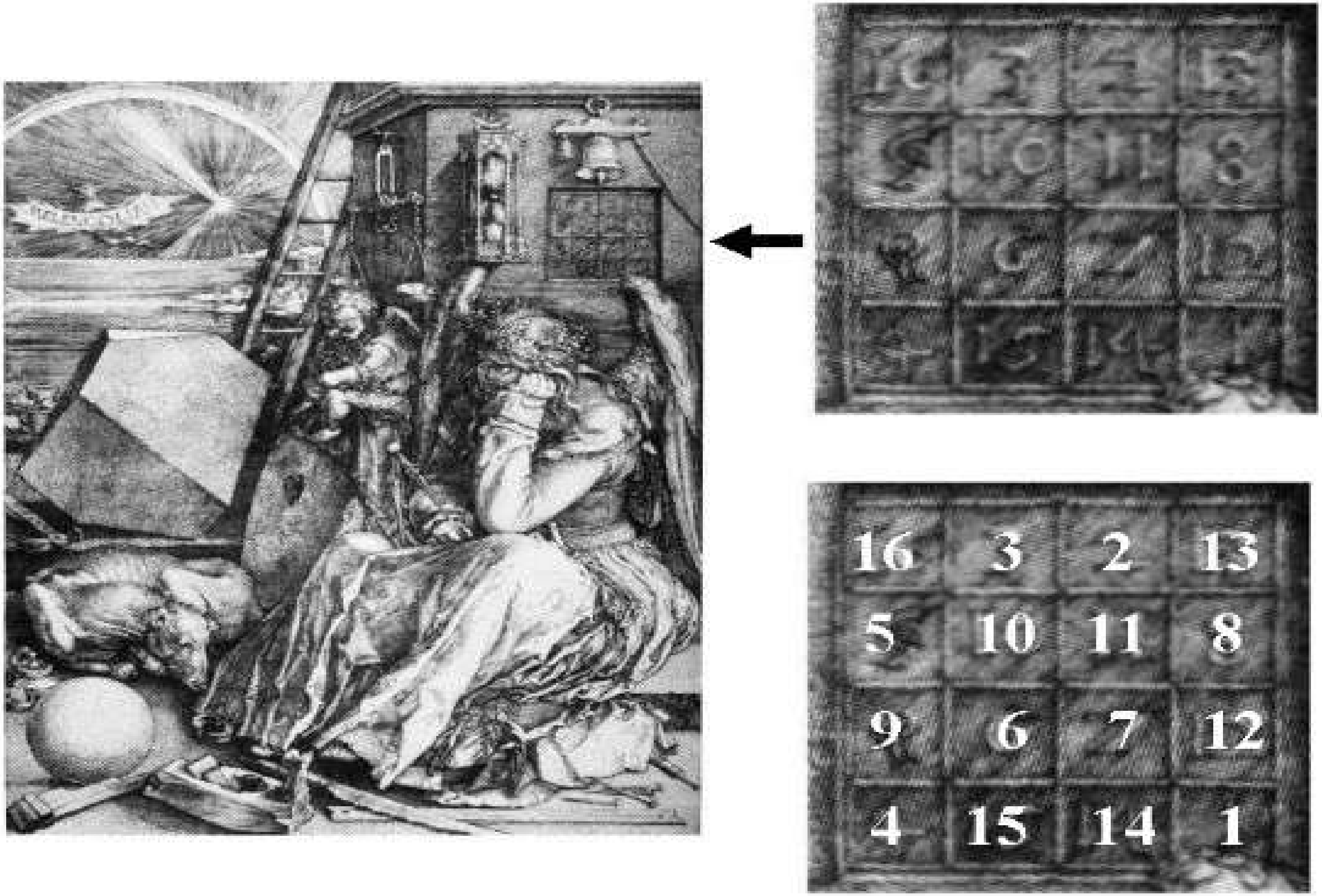
Each of these is a 4×4 Latin square.

A **Latin square** is an $n \times n$ square matrix whose entries consist of n symbols such that each symbol appears exactly once in each row and each column.

Latin Squares

Latin squares have a long history. The concept probably originated with problems concerning the movement and disposition of pieces on a chess board. However, the earliest written reference is the solutions of the card problem published in 1723. The Latin square concept certainly goes back further than this written document. In his famous etching ***Melencholia I***, the 16th Century artist Albrecht Dürer portrays an order 4 magic square, a relative of Latin squares, in the background. Magic squares can also be found in the ancient Chinese literature.

Melencholia I



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Albrecht Dürer's *Melencholia I* (1514)
with Magic Square of order 4.

Latin Squares

The systematic development of Latin squares started with Euler (1779) and was carried on by Cayley (1877-1890) who showed that the multiplication table of a group is an appropriately bordered special Latin square. In the 1930's the concept arose once again in the guise of multiplication tables when the theory of quasi-groups and loops began to be developed as a generalization of the group concept. Latin squares played an important role in the foundations of finite geometries, a subject which was also in development at this time.

Also in the 1930's, a big application area for Latin squares was opened by R.A.Fisher who used them and other combinatorial structures in the design of statistical experiments.

Isotopy

There are three operations that we can perform on a Latin square which will preserve the "Latinness" of the square. They are:

- Permute the rows
- Permute the columns
- Permute the symbols (i.e., rename the symbols without changing their relative positions).

If we can change one square into another by means of any or all of these operations, we say that the two squares are ***isotopic***.

Isotopy is an *equivalence relation* on the set of Latin squares of a given order

Example

The two Latin squares in our solution of the 16 card problem are:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{array} \qquad\qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{array}$$

Note that if we move the second row of the first square into the position of the last row, the two squares would be identical. These squares are isotopic.

Example

On the other hand, these squares are not isotopic!

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array} \qquad\qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array}$$

An ***intercalate*** in a Latin square is a 2×2 subsquare of the form

$$\begin{array}{cc} a & b \\ b & a \end{array}$$

The number of intercalates in a Latin square is not changed by any of the three operations, so isotopic squares have the same number of intercalates. You can count 12 intercalates in the first square, but the second one only contains 4.

Isotopy Classes

Given a square we can permute the columns so that the first row consists of 1 2 3 4 .... n in their natural order. After doing this we could permute the rows so that the first column is also 1 ... n in the natural order. The resulting square is of course isotopic to the original square and is a convenient representative of the isotopy class of this square. Such a square is said to be in ***standard form*** or ***reduced.***

The equivalence classes of Latin squares under the isotopy relation are called ***isotopy classes***. Note that there will in general be more than one reduced square in an isotopy class.

Isotopy Classes

Order	**# Isotopy Classes**	**# Reduced Squares**
2	1	1
3	1	1
4	2	4
5	2	56
6	22	9,408
7	563	16,942,080
8	1,676,257	535,281,401,856
9	**?**	377,597,570,964,258,816

The question of enumerating and classifying Latin squares is not an easy one. Give some thought to how you might attempt to do this.

Orthogonality

The property needed by the pair of Latin squares in the solution of the 16 card problem is called **orthogonality**.

Two Latin squares $L_1 = | a_{ij}|$ and $L_2 = | b_{ij}|$ on n symbols 1,2,...,n are said to be ***orthogonal*** if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}), i = 1,2,...,n; j = 1,2,...,n.

For example, these squares are orthogonal:

$$
\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}
\qquad
\begin{array}{ccc} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{array}
\qquad\qquad
\begin{array}{ccc} (2,2) & (3,1) & (1,3) \\ (1,1) & (2,3) & (3,2) \\ (3,3) & (1,2) & (2,1) \end{array}
$$

Orthogonal Mates

We often refer to one of a pair of orthogonal squares as being an ***orthogonal mate*** of the other.

Euler was originally interested in such pairs and in his writings he would always use Latin letters for the first square and Greek letters for the second. Thus, when he referred to the first of the squares he referred to the ***Latin square***.

When referring to both of the orthogonal squares he used the term ***Graeco-Latin squares***, which is the way orthogonal squares are referred to in all the earlier literature.

Transversals

Given a pair of orthogonal Latin squares, consider the cells in the first square which contain one particular symbol. Of course, by “Latinness”, there is only one of these cells in each row and column. Now consider the cells in the orthogonal mate which correspond to these cells in the first square. By orthogonality, the entries in these cells must all be different and so these cells form a ***transversal*** in the orthogonal mate. This observation applies to the cells corresponding to any symbol of the first square, so we have:

Theorem - *A given Latin square possesses an orthogonal mate if and only if it has n disjoint transversals.*

Odd Order

If the Latin square in question is the multiplication table of a group of odd order n, then it can be shown that the existence of a single transversal implies the existence of n disjoint transversals, so we conclude that the square has an orthogonal mate.

Theorem - *The multiplication table of any group of odd order forms a Latin square which possesses an orthogonal mate.*

Corollary - *There exist pairs of orthogonal Latin squares of every odd order.*

Even Order

The existence question for pairs of orthogonal Latin squares of even order is much more difficult to settle and has a long and famous history. To start with, there are only two Latin squares of order 2 and they are not orthogonal. We have given an example of a pair of orthogonal squares of order 4. The next case, that of order 6, is the problem that originally interested Euler in the subject. Called the ***Problem of the 36 Officers***, Euler stated it as follows in 1779, "Arrange 36 officers, 6 from each of 6 regiments, of 6 different ranks, into a 6x6 square, so that each row and each file contains one officer of each rank and one officer of each regiment." Clearly, the problem can be solved, as with the card problem, by finding a pair of orthogonal latin squares of order 6.

Euler's Conjecture

As brilliant a mathematician as Euler was, he was unable to find such a pair of squares and unable to prove that they did not exist. Based on his experience with the problem and some other pieces of evidence (such as the Corollary, which he was aware of), Euler made a conjecture which included and went beyond this problem:

Euler's Conjecture: *There does not exist an orthogonal mate for any Latin square whose order has the form* $n = 4k + 2$ (oddly even integers as he put it).

Tarry's Result

120 years after Euler first stated the problem, Tarry in 1900 settled the problem of the 36 officers in the negative. His method was straight-forward, he listed out all of the 812,851,200 Latin squares of order 6 and examined each pair for orthogonality and found none [actually, by working with reduced squares he simplified the problem to checking only 9408 pairs – but of course this was all done by hand].

It was beginning to look like the old master had pulled off a coupe, but in 1960 Bose, Shrikhande and Parker shocked the mathematical community by proving that if $n > 6$ Euler's conjecture is false. Their original method is long, complicated and involved, looking at a number of special cases, but has since been simplified.

MOLS

A set of Latin squares of the same order, each of which is an orthogonal mate of each of the others is called a ***set of mutually orthogonal Latin squares***. This mouthful is often shortened to its acronym **MOLS**. For example, a set of 3 MOLS of order 4 is given by:

$$\begin{array}{cccc} 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{cccc} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{cccc} 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{array}$$

Question: How large can a set of MOLS be?

Equivalent MOLS

Two sets of MOLS with the same number of squares are said to be ***equivalent sets of MOLS*** if one can be obtained from the other by any combination of simultaneously permuting the rows of all the squares, simultaneously permuting the columns of all the squares and renaming the elements of any square.

Lemma - *Any set of MOLS is equivalent to a set where each square has the first row in natural order and one of the squares (usually the first) is reduced (i.e., it also has its first column in natural order).*

Pf: Given a set of MOLS, we can convert it to an equivalent set by renaming the elements in each square so that the first rows are all in natural order. Now take any square and simultaneously permute the rows of all the squares so that the first column of this square is in natural order (this will not affect the first row since it is in natural order and so starts with the smallest element). The result is an equivalent set with the required properties.

Number of MOLS

A set of MOLS in the form described by this lemma is said to be in ***standard form***, and the lemma merely says that any set of MOLS is equivalent to a set of MOLS in standard form.

Theorem - *No more than n-1 MOLS of order n can exist.*

Pf: Any set of MOLS of order n is equivalent to a set in standard form, which has the same number of squares in it. Consider the entries in first column and second row of all of the squares in standard form. No two squares can have the same entry in this cell. Suppose two squares had an r, say, in this cell, then in the superimposed square the ordered pair (r,r) would appear in this cell and also in the r-th cell of the first row because both squares have the same first row, and so, the two squares can not be orthogonal contradiction. Now, we can not have a 1 in this cell, since it appears in the first column of the first row. Thus, there are only n-1 possible entries for this cell and so there can be at most n-1 squares.

Complete Sets of MOLS

A set of MOLS of order n containing n-1 squares is called a ***complete set***. We now have an existence question, for which orders do complete sets of MOLS exist?

We know by examples that complete sets exist for orders 2 (only one square is needed), 3 and 4 and also that no complete set exists for orders 6 and 10. Using finite fields, complete sets of MOLS for any order which is a prime or power of a prime can be constructed. However, it is an open research question of long standing whether or not a complete set of MOLS exists for any composite order.

Instant fame will go to any mathematician who can settle this question.