

Combinatorial Designs: Balanced Incomplete Block Designs

Designs

The theory of design of experiments came into being largely through the work of R.A.Fisher and F.Yates in the early 1930's. They were motivated by questions of design of field experiments in agriculture. The applicability of this theory is now very widespread, much of the terminology still bears the stamp of its origins.

Consider an agricultural experiment. Suppose it is desired to compare the yield of v different varieties of grain. It is quite possible that there would be an interaction between the environment (type of soil, rainfall, drainage, etc.) and the variety of grain which would alter the yields. So, b blocks (sets of experimental plots) are chosen in which the environment is fairly consistent throughout the block. In other types of experiments, in which the environment might not be a factor, blocks could be distinguished as plots which receive a particular treatment (say, are given a particular type of fertilizer). In this way, the classification of the experimental plots into blocks and varieties can be used whenever there are two factors which may influence yield.

Designs

The obvious technique of growing every variety in a plot in every block, may, for large experiments be too costly or impractical. To deal with this, one would use smaller blocks which did not contain all of the varieties. Now the problem is one of comparison, to minimize the effects of chance due to incomplete blocks, we would want to design the blocks so that the probability of two varieties being compared (i.e. are in the same block) is the same for all pairs. This property would be called *balance* in the design. Statistical techniques, in particular Analysis of Variance (ANOVA), could then be used to reach conclusions about the experiment.

BIBD's

A **BIBD** is a set X of $v \geq 2$ elements called *varieties* or *treatments* and a collection of $b > 0$ subsets of X , called *blocks*, such that the following conditions are satisfied:

- each block consists of exactly k varieties, $v > k > 0$,
- each variety appears in exactly r blocks, $r > 0$,
- each pair of varieties appear simultaneously in exactly λ blocks, $\lambda > 0$.

BIBD's

The incomplete in the name of these designs refers to the condition $v > k$, i.e., the block size, k , is less than the total number of treatments, so no block contains all the varieties. If we allowed $v = k$, then all the conditions would be trivially satisfied and the resulting design would not be of much interest. Balanced refers to the constancy of the λ parameter. BIBD's are often referred to as (v,b,r,k,λ) - designs.

An example of a $(7,7,3,3,1)$ -design is given by the set X consisting of the varieties, 1,2,3,4,5,6,7 and the following blocks:
 $\{1,2,4\}$ $\{2,3,5\}$ $\{3,4,6\}$ $\{4,5,7\}$ $\{5,6,1\}$ $\{6,7,2\}$ $\{7,1,3\}$.

It is easily seen that the block size k is 3, and the repetition number r is also 3. Each pair of varieties appears together in only one block, so $\lambda = 1$.

Other Examples

A $(4,4,3,3,2)$ -design is given by $X = \{1,2,3,4\}$ with blocks:

$$\{1,2,3\} \quad \{2,3,4\} \quad \{3,4,1\} \quad \{4,1,2\}.$$

A slightly larger example is the $(8,14,7,4,3)$ -design on the set $X = \{1,2,3,4,5,6,7,8\}$ with blocks:

$$\begin{aligned} &\{1,3,7,8\} \quad \{1,2,4,8\} \quad \{2,3,5,8\} \quad \{3,4,6,8\} \quad \{4,5,7,8\} \quad \{1,5,6,8\} \\ &\{2,6,7,8\} \quad \{1,2,3,6\} \quad \{1,2,5,7\} \quad \{1,3,4,5\} \quad \{1,4,6,7\} \quad \{2,3,4,7\} \\ &\{2,4,5,6\} \quad \{3,5,6,7\}. \end{aligned}$$

Parameter Conditions

Some elementary counting arguments show that there are necessary conditions that the parameters of a BIBD must satisfy.

Theorem - Given a (v,b,r,k,λ) -design,

- $bk = vr$
- $r(k-1) = \lambda(v-1)$.

Proof: Consider the set of pairs (x,B) , where x is a variety and B is a block containing x . By counting this set in two ways we arrive at the first equation. There are v possible values for x , and since each appears in r blocks, vr will count the number of these pairs. On the other hand, there are b blocks and each contains k varieties, so bk also counts the number of these pairs.

Parameter Conditions

Theorem - Given a (v,b,r,k,λ) -design,

- $bk = vr$
- $r(k-1) = \lambda(v-1)$.

Proof (cont.): The second equation is also obtained by counting. Fix a particular variety, say p , and count the number of pairs of varieties $\{p,y\}$ where p and y appear in some block together and if the pair appears more than once it is multiply counted. There are $v - 1$ possible choices of y and each such pair will appear in λ blocks together, so there are $\lambda(v-1)$ such pairs. On the other hand p appears in r blocks and can be paired with $k - 1$ other elements in such a block, thus $r(k-1) = \lambda(v-1)$.

Incidence Matrix

Given a (v,b,r,k,λ) -design, we can represent it with a $v \times b$ matrix called the *incidence matrix* of the design. The rows are labeled with the varieties of the design and the columns with the blocks. We put a 1 in the (i,j) -th cell of the matrix if variety i is contained in block j and 0 otherwise. Each row of the incidence matrix has r 1's, each column has k 1's and each pair of distinct rows has λ common 1's. These observations lead to a useful matrix identity.

Example

For the design $\{1,2,3\}$ $\{2,3,4\}$ $\{3,4,1\}$ $\{4,1,2\}$ we construct the incidence matrix.

| | $\{1,2,3\}$ | $\{2,3,4\}$ | $\{3,4,1\}$ | $\{4,1,2\}$ |
|---|-------------|-------------|-------------|-------------|
| 1 | 1 | 0 | 1 | 1 |
| 2 | 1 | 1 | 0 | 1 |
| 3 | 1 | 1 | 1 | 0 |
| 4 | 0 | 1 | 1 | 1 |

Matrix Identity for Designs

Theorem - If \mathbf{A} is the incidence matrix of a (v,b,r,k,λ) -design, then

$$\mathbf{A}\mathbf{A}^T = (r-\lambda)\mathbf{I} + \lambda\mathbf{J}$$

where \mathbf{I} is the $v \times v$ identity matrix and \mathbf{J} is the $v \times v$ matrix of all 1's.

Proof: In the product, an off diagonal entry is the inner product of two distinct rows of \mathbf{A} , which must be λ . A diagonal entry is the inner product of a row of \mathbf{A} with itself, and so equals r .

Fisher's Inequality

Theorem - [Fisher's Inequality] *In a (v,b,r,k,λ) -design, $b \geq v$.*

Proof: Suppose that $b < v$ and let \mathbf{A} be the incidence matrix of the design. We can add $v - b$ columns of 0's to \mathbf{A} to get a $v \times v$ matrix \mathbf{B} . Since these extra columns of 0's will not alter the inner products, we must have $\mathbf{A}\mathbf{A}^T = \mathbf{B}\mathbf{B}^T$. By taking determinants we see that,

$$\det(\mathbf{A}\mathbf{A}^T) = \det(\mathbf{B}\mathbf{B}^T) = (\det \mathbf{B})(\det \mathbf{B}^T) = 0$$

since $\det \mathbf{B} = 0$ due to the columns of 0's. Now by the matrix identity we have:

$$\det(\mathbf{A}\mathbf{A}^T) = \det \begin{pmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{pmatrix}.$$

Fisher's Inequality

Theorem - [Fisher's Inequality] *In a (v,b,r,k,λ) -design, $b \geq v$.*

Pf (cont) : We can evaluate this determinant by subtracting the first column from each of the other columns and then adding each row to the first row to obtain the following:

$$\det(AA^T) = \det \begin{pmatrix} r + \lambda(v-1) & 0 & 0 & \cdots & 0 \\ \lambda & r - \lambda & 0 & \cdots & 0 \\ \lambda & 0 & r - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & \cdots & r - \lambda \end{pmatrix}.$$

So we have that,

$$\det(\mathbf{AA}^T) = [r + \lambda(v-1)](r-\lambda)^{v-1}.$$

But since r, v and λ are all positive, $r + \lambda(v-1) > 0$ and since $k < v$ we have that $r > \lambda$, so this product on the right can not equal 0

contradiction.

Symmetric Designs

If in a BIBD we have $v = b$ (and thus $r = k$), we say that the BIBD is *symmetric* or *square* (maybe, *projective*). Symmetric BIBD's are often referred to as (v, k, λ) -designs. For symmetric BIBD's, there is an additional constraint on the parameters.

Bruck-Ryser-Chowla Theorem: *The following conditions are necessary for the existence of a symmetric BIBD:*

- 1. If v is even, then $k - \lambda$ is the square of an integer.*
- 2. If v is odd, then the following equation has a solution in integers x, y, z not all of which are 0:*

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2} \lambda z^2$$

The proof of this result involves some very deep results from number theory and so we will not give it.

Parameters

All of the results on the parameters of a BIBD that we have given have been necessary but not sufficient. That is, we can use them to rule out the existence of a BIBD for certain sets of parameters, but given a set of the parameters which satisfy all these conditions does not mean that there actually exists a BIBD with those parameters. There are many sets of possible parameters for which the existence question has not been settled.

Thus, a symmetric $(111, 11, 1)$ -design does not exist even though these parameters satisfy all the conditions that we have given.

Some Families of BIBD's

$$k = 2, \lambda = 1$$

If a BIBD has the parameters $k = 2$ and $\lambda = 1$, then it is easily calculated that $r = v - 1$ and $b = v(v - 1)/2$. This means that the blocks of the design are just all possible pairs of varieties, i.e., the set of blocks is the set of all 2-subsets of X .

If we interpret the varieties of the design as being vertices and the blocks as being edges, then a design with these parameters is a complete graph on v vertices. From the viewpoint of design theory, these are not very interesting designs even though they are an important class of graphs.

Some Families of BIBD's

When $k = 3$ and $\lambda = 1$, the BIBD is called a **Steiner Triple System** since the blocks are triples and each pair of varieties appears in exactly one triple.

A single triple contains 3 pairs and there are a total of $v(v-1)/2$ pairs of elements. Since each pair appears in exactly one triple,

$$3b = v(v-1)/2$$

so, $b = v(v-1)/6$.

Now consider one element x . Every other element is paired with x in some block, but these blocks contain two other elements with no repeats, so $r = (v-1)/2$, and since r is an integer, v must be odd.

Since b is an integer and v is odd, either 3 divides v or 6 divides $v-1$, so $v \equiv 3 \pmod{6}$ or $v \equiv 1 \pmod{6}$.

Steiner Triple Systems

Examples:

$$v = 1 \quad \emptyset$$

$$v = 3 \quad \{1,2,3\}$$

$$v = 7 \quad \{1,2,4\} \quad \{2,3,5\} \quad \{3,4,6\} \quad \{4,5,7\} \quad \{5,6,1\} \quad \{6,7,2\} \quad \{7,1,3\}$$

$$v = 9 \quad \{1,2,3\} \quad \{4,5,6\} \quad \{7,8,9\} \quad \{1,4,7\} \quad \{2,5,8\} \quad \{3,6,9\} \quad \{1,5,9\} \\ \{2,6,7\} \quad \{3,4,8\} \quad \{1,6,8\} \quad \{2,4,9\} \quad \{3,5,7\}$$

Steiner Triple Systems

This necessary condition is also sufficient for the existence of a Steiner Triple System.

Theorem - [Kirkman, 1847] *There exists a Steiner Triple System of v varieties iff $v \geq 3$ and $v \equiv 1$ or $3 \pmod{6}$.*

Proof: We have already shown the necessity of these conditions. The sufficiency is proved by construction, but as this is rather long and messy we will not present it [details may be found in M. Hall's, *Combinatorial Theory*].

Kirkman's Schoolgirl Problem

In 1853 J. Steiner posed the sufficiency of this theorem as a problem and it was proved in 1859 by M. Reiss. Neither mathematician was aware of the fact that the problem had been posed and solved by T.P. Kirkman in an 1847 article appearing in the Cambridge and Dublin Mathematical Journal. Indeed, in 1850 Kirkman went on to pose a more difficult but related problem. This problem, which appeared in "The Lady's and Gentleman's Diary" of 1850, has become to be known as Kirkman's Schoolgirl Problem and was presented as follows:

A teacher would like to take 15 schoolgirls out for a walk, the girls being arranged in 5 rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence it is desirable to find different row arrangements for the 7 days of the week such that any pair of girls walk in the same row exactly one day of the week.

Kirkman's Schoolgirl Problem

This problem, which in general asks for a Steiner Triple System on $6t + 3$ varieties whose blocks can be partitioned into $3t + 1$ sets so that any variety appears only once in a set, had not been settled in general until 1971 (Ray-Chaudhuri and Wilson).

A solution of the schoolgirl problem:

| | | | | | | |
|---------|---------|---------|---------|---------|---------|---------|
| 1 2 3 | 1 4 5 | 1 6 7 | 1 8 9 | 1 10 11 | 1 12 13 | 1 14 15 |
| 4 8 12 | 2 8 10 | 2 9 11 | 2 12 15 | 2 13 14 | 2 4 6 | 2 5 7 |
| 5 10 14 | 3 13 15 | 3 12 14 | 3 5 6 | 3 4 7 | 3 9 10 | 3 8 11 |
| 6 11 13 | 6 9 14 | 4 10 15 | 4 11 14 | 5 9 12 | 5 11 15 | 4 9 13 |
| 7 9 15 | 7 11 12 | 5 8 13 | 7 10 13 | 6 8 15 | 7 8 14 | 6 10 12 |

Some Families of BIBD's

Symmetric designs with $\lambda = 1$ are known as *finite projective planes*. The **order** of a design is $k - \lambda$.

While we won't prove it here, a finite projective plane of order n is equivalent to a complete set of Latin squares of order n .

Some existence results for finite projective planes are given in the following table:

Finite Projective Planes

Finite Projective Planes of Small Order

| | | | | | | | | | | | | | | | | | | | |
|-------|---|---|---|---|---|---|---|---|----|----------|----|----------|----|----|-----------|----------|----|----------|----|
| Order | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| # | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 4 | 0 | ≥ 1 | ?? | ≥ 1 | 0 | ?? | ≥ 22 | ≥ 1 | ?? | ≥ 1 | ?? |

Notes:

1. The only *known* projective planes have orders that are prime powers.
2. (*Bruck-Ryser Theorem*) If $q \equiv 1$ or $2 \pmod{4}$ and a projective plane of order q exists, then q is the sum of two integral squares (one of which may be 0).
3. The order 10 was eliminated by a massive computer search.

Finite Projective Planes

Finite Projective Planes of Small Order

| | | | | | | | | | | | | | | | | | | | |
|-------|---|---|---|---|---|---|---|---|----|----------|----|----------|----|----|-----------|----------|----|----------|----|
| Order | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| # | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 4 | 0 | ≥ 1 | ?? | ≥ 1 | 0 | ?? | ≥ 22 | ≥ 1 | ?? | ≥ 1 | ?? |

Notes:

4. For each prime or prime power there is a standard construction that gives a projective plane of that order.
5. For prime orders, only the above construction is known to give a plane. It is conjectured that there are no others.
6. The fact that there are exactly 4 projective planes of order 9 was established by a large computer search.

Some Families of BIBD's

A symmetric design with $\lambda = 2$ is called a *biplane*.

Whereas we can construct an infinite number of finite projective planes, only a finite number of biplanes are known. These $(v,k,2)$ designs exist for $k = 2, 3, 4, 5, 6, 9, 11$ and 13 .

Examples:

$k = 2$: $\{1,2\} \{1,2\}$

$k = 3$: $\{1,2,3\} \{2,3,4\} \{3,4,1\} \{4,1,2\}$.

$k = 4$: $\{3,5,6,7\} \{4,6,7,1\} \{5,7,1,2\} \{6,1,2,3\}$
 $\{7,2,3,4\} \{1,3,4,5\} \{2,4,5,6\}$

Biplanes

As a consequence of the Bruck-Ryser-Chowla Theorem:

Theorem: For a biplane

(i) if $k = 2$ or $3 \pmod{4}$, then v is even and $k-2$ is a square.

(ii) if $k = 0$ or $1 \pmod{4}$, then

$$z^2 = (k-2) x^2 + (-1)^{(v-1)/2} 2y^2$$

has a solution in integers not all zero.

Biplanes

It is easy to see that the values $k = 7, 8, 10$ and 12 are eliminated by this theorem.

$k = 7$ and 10 are eliminated since 5 and 8 are not squares.

For $k = 8$, we calculate that $v = 29$ and examine the equation

$$z^2 = 6x^2 + 2y^2.$$

If there is a solution, there is a solution with x, y and z being relatively prime, so assume this is so. Clearly, in a solution, z must be even, so let $z = 2k$ and rewrite the equation as

$$2k^2 = 3x^2 + y^2.$$

Consider this equation modulo 3 , $2k^2 \equiv y^2 \pmod{3}$. If k is not a multiple of 3 , then $2k^2 \equiv 2 \pmod{3}$, but 2 is not a square mod 3 . Thus 3 divides k which forces 3 to divide y . Returning to the original equation we see that $9 \mid 6x^2$, so 3 must divide x . $\rightarrow\leftarrow$

Biplanes

The biplanes with $k = 2, 3, 4$ and 5 are unique.

There are 3 biplanes with $k = 6$ (up to isomorphism).

There are 4 biplanes with $k = 9$ (computer search)

There are 5 known biplanes with $k = 11$.

There are 2 known biplanes with $k = 13$.

Can you find any more?