

Quadratic Sets

4.7 Quadrics

Quadratic Forms

Def: Let V be a vector space over a field F . A map $q: V \rightarrow F$ is called a *quadratic form* of V if

i) $q(av) = a^2q(v)$ for all v in V and a in F .

ii) the map $B: V \times V \rightarrow F$ defined by

$$B(v,w) = q(v+w) - q(v) - q(w)$$

is a symmetric bilinear form.

Examples

Consider the 3 dimensional vector space V over the reals.

$$q(x,y,z) = x^2 + y^2 + z^2$$

is a quadratic form.

$$\begin{aligned} q(ax,ay,az) &= (ax)^2 + (ay)^2 + (az)^2 = a^2(x^2 + y^2 + z^2) \\ &= a^2q(x,y,z). \end{aligned}$$

With $v = (x,y,z)$ and $w = (x',y',z')$ we have

$$\begin{aligned} B(v,w) &= (x+x')^2 + (y+y')^2 + (z+z')^2 - (x^2+y^2+z^2) - (x'^2 + y'^2 + z'^2) \\ &= 2xx' + 2yy' + 2zz'. \end{aligned}$$

Clearly, $B(v,w) = B(w,v)$ so this form is symmetric.

Let $v' = (a,b,c)$, then

$$\begin{aligned} B(v+v',w) &= 2(x+a)x' + 2(y+b)y' + 2(z+c)z' \\ &= 2xx' + 2yy' + 2zz' + 2ax' + 2by' + 2cz' = B(v,w) + B(v',w). \end{aligned}$$

$$B(dv,w) = 2dxx' + 2dyy' + 2dzz' = d B(v,w).$$

Thus B is linear in the first coordinate, and by symmetry is bilinear.

Examples

Consider the 3 dimensional vector space V over the reals.

$$q(x,y,z) = xy - z^2$$

is a quadratic form.

$$\begin{aligned} q(ax,ay,az) &= (ax)(ay) - (az)^2 = a^2(xy - z^2) \\ &= a^2q(x,y,z). \end{aligned}$$

With $v = (x,y,z)$ and $w = (x',y',z')$ we have

$$\begin{aligned} B(v,w) &= (x+x')(y+y') - (z+z')^2 - (xy - z^2) - (x'y' - z'^2) \\ &= xy' + x'y - 2zz'. \end{aligned}$$

Clearly, $B(v,w) = B(w,v)$ so this form is symmetric.

Let $v' = (a,b,c)$, then

$$\begin{aligned} B(v+v',w) &= (x+a)y' + x'(y+b) - 2(z+c)z' \\ &= xy' + x'y - 2zz' + ay' + x'b - 2cz' = B(v,w) + B(v',w). \end{aligned}$$

$$B(dv,w) = dxy' + x'dy - 2dzz' = d B(v,w).$$

Thus B is linear in the first coordinate, and by symmetry is bilinear.

Standard Form

Lemma 4.7.1: Let $\{v_1, \dots, v_n\}$ be a basis of the vector space V .

a) If a_{ij} in F , then a quadratic form is defined by the following rule:

$$q\left(\sum_{i=1}^n b_i v_i\right) = \sum_{i,j=1}^n a_{ij} b_i b_j.$$

b) Conversely, for any quadratic form q there are elements a_{ij} in F such that q has the above form.

Pf: For any a in F we have

$$\begin{aligned} q(av) &= q\left(a \sum_{i=1}^n b_i v_i\right) = q\left(\sum_{i=1}^n (ab_i) v_i\right) = \sum_{i,j=1}^n a_{ij} (ab_i) (ab_j) \\ &= a^2 \sum_{i,j=1}^n a_{ij} b_i b_j = a^2 q(v). \end{aligned}$$

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Pf(cont.): Consider $B(v,w) = q(v+w) - q(v) - q(w)$.

$$\begin{aligned} &= \sum_{i,j=1}^n a_{ij} (b_i + c_i)(b_j + c_j) - \sum_{i,j=1}^n a_{ij} b_i b_j - \sum_{i,j=1}^n a_{ij} c_i c_j \\ &= \sum_{i,j=1}^n a_{ij} (b_i c_j + b_j c_i). \end{aligned}$$

From this it is easy to see that $B(v,w) = B(w,v)$, $B(v+v',w) = B(v,w) + B(v',w)$, and $B(v, w+w') = B(v,w) + B(v,w')$, i.e., B is a symmetric bilinear form.

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*Pf(cont.):*b) Let q be a quadratic form and $v = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$.

$$\begin{aligned} q(v) &= q\left(b_1 v_1 + \sum_{i=2}^n b_i v_i\right) = q(b_1 v_1) + q\left(\sum_{i=2}^n b_i v_i\right) + B\left(b_1 v_1, \sum_{i=2}^n b_i v_i\right) \\ &= b_1^2 q(v_1) + q\left(\sum_{i=2}^n b_i v_i\right) + \sum_{i=2}^n B(b_1 v_1, b_i v_i). \end{aligned}$$

Now repeat this decomposition with the second term on the right to get:

Standard Form

Lemma 4.7.1: Let $\{v_1, \dots, v_n\}$ be a basis of the vector space V .

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b) Conversely, for any quadratic form q there are elements a_{ij} in F such that q has the above form.

Pf(cont.):

$$q(v) = b_1^2 q(v_1) + b_2^2 q(v_2) + q\left(\sum_{i=3}^n b_i v_i\right) + \sum_{i=2}^n B(b_1 v_1, b_i v_i) + \sum_{i=3}^n B(b_2 v_2, b_i v_i).$$

Continue this procedure to obtain:

$$q(v) = \sum_{i=1}^n b_i^2 q(v_i) + \sum_{i < j} B(b_i v_i, b_j v_j) = \sum_{i=1}^n b_i^2 q(v_i) + \sum_{i < j} b_i b_j B(v_i, v_j).$$

Standard Form

Lemma 4.7.1: Let $\{v_1, \dots, v_n\}$ be a basis of the vector space V .

a) If a_{ij} in F , then a quadratic form is defined by the following rule:

$$q\left(\sum_{i=1}^n b_i v_i\right) = \sum_{i,j=1}^n a_{ij} b_i b_j.$$

b) Conversely, for any quadratic form q there are elements a_{ij} in F such that q has the above form.

Pf(cont.): We see then that q has the required form when we pick $a_{ii} = q(v_i)$, $a_{ij} = B(v_i, v_j)$ if $i < j$ and $a_{ij} = 0$ if $i > j$. \square

Quadrics

Def: A quadratic form is *nondegenerate* if $q(v) = 0$ and $B(x, v) = 0$ for all x in V implies that $v = 0$.

Def: Let q be a quadratic form of the vector space V . The *quadric* of the projective space $P(V)$ corresponding to q is the set of all points $\langle v \rangle$ of $P(V)$ with $q(v) = 0$.

Examples

By Theorem 2.4.4 the points on a regulus in a 3-dimensional projective space can be given coordinates that satisfy,

$$x_0 x_3 - x_1 x_2 = 0.$$

Since $x_0 x_3 - x_1 x_2$ is a quadratic form, these points lie on a quadric (the hyperbolic quadric).

The quadratic form $x_0^2 - x_1^2 - x_2^2$ yields a quadric in the plane which is an oval. The quadratic form $x_0^2 - x_1^2 + x_2^2$ yields a quadric in the plane which is the union of two lines.

In a 3-dimensional projective space, the quadric given by $x_0^2 - x_1^2 - x_2^2 - x_3^2$ is an ovoid, while the quadric given by $x_0^2 + x_1^2 - x_2^2 - x_3^2$ is a hyperboloid.

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Lemma 4.7.2: Let q be a quadratic form of a vector space V , and let Q be the corresponding quadric in $P(V)$. Then, if a line g contains three points of Q , each point of g lies in Q .

Pf: Let $P = \langle v \rangle$, $R = \langle w \rangle$ and S be three different points on Q . We may assume that $S = \langle v + aw \rangle$ with $a \neq 0$. Then since S in Q ,

$$\begin{aligned} 0 &= q(v + aw) = B(v, aw) + q(v) + q(aw) \\ &= aB(v, w) + q(v) + a^2q(w) = aB(v, w). \end{aligned}$$

Since $a \neq 0$, $B(v, w) = 0$. Now, for each point $X = \langle v + bw \rangle \neq R$ on g we have

$$\begin{aligned} q(v+bw) &= B(v, bw) + q(v) + q(bw) \\ &= bB(v, w) + q(v) + b^2q(w) \\ &= bB(v, w) = 0. \end{aligned}$$

Hence each point of g is in Q . □

Perps

Def: Let q be a quadratic form of the vector space V . For a non-zero vector v in V we define

$$\langle v \rangle^\perp = \{x \in V \mid B(x,v) = 0 \}.$$

Tangents

Lemma 4.7.3: Let q be a quadratic form of the vector space V , and let Q be the corresponding quadric of $\mathbf{P}(V)$. Then for each non-zero vector v in V we have:

- a) $\langle v \rangle^\perp$ is a subspace of V , hence also a subspace of $\mathbf{P}(V)$.
- b) $\langle v \rangle^\perp$ is either a hyperplane or equal to V .
- c) Suppose that $\langle v \rangle$ is in Q . Then each line of $\langle v \rangle^\perp$ through $\langle v \rangle$ is a tangent line of Q .
- d) Suppose that $\langle v \rangle$ is not in Q . Then each line through $\langle v \rangle$ that is not contained in $\langle v \rangle^\perp$ intersects Q in precisely one further point.

Pf: a) follows from the fact that B is a bilinear form.

b) will follow if we can show that every line $g = \langle u, w \rangle$ of $\mathbf{P}(V)$ intersects the subspace $\langle v \rangle^\perp$ in at least one non-zero vector. We may assume that $\langle w \rangle \notin \langle v \rangle^\perp$. Thus, $B(w, v) \neq 0$.

Tangents

Lemma 4.7.3: Let q be a quadratic form of the vector space V , and let Q be the corresponding quadric of $\mathbf{P}(V)$. Then for each non-zero vector v in V we have:

- a) $\langle v \rangle^\perp$ is a subspace of V , hence also a subspace of $\mathbf{P}(V)$.
- b) $\langle v \rangle^\perp$ is either a hyperplane or equal to V .
- c) Suppose that $\langle v \rangle \in Q$. Then each line of $\langle v \rangle^\perp$ through $\langle v \rangle$ is a tangent line of Q .
- d) Suppose that $\langle v \rangle \notin Q$. Then each line through $\langle v \rangle$ that is not contained in $\langle v \rangle^\perp$ intersects Q in precisely one further point.

Pf (cont.): Let $X = \langle u+aw \rangle$ be an arbitrary point of g different from $\langle w \rangle$.

Then $X \in \langle v \rangle^\perp$

$$\Leftrightarrow B(u+aw, v) = 0 \Leftrightarrow B(u, v) + aB(w, v) = 0 \Leftrightarrow a = -B(u, v)/B(w, v).$$

c) Let $\langle v+aw \rangle$ be an arbitrary point $\neq \langle w \rangle$ on the line $g = \langle v, w \rangle$. Then $q(v+aw) = B(v, aw) + q(v) + q(aw) = aB(v, w) + q(v) + a^2q(w) = a^2q(w)$. Thus, if $\langle w \rangle \in Q$ all points of g are in Q , and if $\langle w \rangle \notin Q$ only the point $\langle v \rangle$ is in Q . So, g is a tangent line of Q .

Tangents

Lemma 4.7.3: Let q be a quadratic form of the vector space V , and let Q be the corresponding quadric of $\mathbf{P}(V)$. Then for each non-zero vector v in V we have:

- a) $\langle v \rangle^\perp$ is a subspace of V , hence also a subspace of $\mathbf{P}(V)$.
- b) $\langle v \rangle^\perp$ is either a hyperplane or equal to V .
- c) Suppose that $\langle v \rangle \in Q$. Then each line of $\langle v \rangle^\perp$ through $\langle v \rangle$ is a tangent line of Q .
- d) Suppose that $\langle v \rangle \in Q$. Then each line through $\langle v \rangle$ that is not contained in $\langle v \rangle^\perp$ intersects Q in precisely one further point.

Pf (cont.): d) Suppose that $\langle w \rangle \in \mathbf{P} \setminus \langle v \rangle^\perp$ and let $\langle w+av \rangle$ be an arbitrary point on the line $\langle v, w \rangle$ that is different from $\langle v \rangle$. Then

$$\begin{aligned} q(w+av) &= B(w, av) + q(w) + q(av) \\ &= aB(w, v) + q(w). \end{aligned}$$

If $q(w+av) = 0$ it follows that $a = -q(w)/B(w, v)$. Hence the line $\langle v, w \rangle$ intersects the quadric Q only at $\langle v \rangle$ and $\langle w+av \rangle$. \square

Quadratic Sets

Theorem 4.7.4: Each quadric is a quadratic set.

Pf: By 4.7.2 the if three then all axiom holds. By Lemma 4.7.3 for any point $\langle v \rangle$ of a quadric, the subspace $\langle v \rangle^\perp$ is the corresponding tangent hyperplane. \square

Buekenhout's Theorem

A fundamental theorem of Buekenhout states that any non-empty nondegenerate quadratic set is either a quadric or an *ovaloid* (a non-empty nondegenerate quadratic set with the property that it contains no three collinear points, i.e., has index 1). We have already proved this for hyperbolic sets in 3-dimensional projective spaces (**by showing that these non-ovaloids are quadrics.**) In the next section we prove an analogous statement for Klein quadratic sets.

Segre's Theorem

In the case $d = 2$ the an improvement was first proved by B. Segre.

Theorem 4.7.5: Any oval in a finite Desarguesian projective plane of odd order is a conic (a nonempty, nondegenerate quadric in a projective plane of odd order is a quadric).

In the case $d = 3$ the corresponding result was first given by Barlotti and Panella, who show that in a 3-dimensional projective space of odd order, every non-degenerate quadratic set is a quadric.

The ovaloids in even order planes and 3-spaces have not been classified – this is a very difficult research problem.

Klein Quadratic Sets

By means of a specialized coordinatization (Plücker coordinates) each line of a 3 dimensional projective space can be associated with a point of a 5 dimensional projective space in such a way that the points in the 5 dimensional projective space form a hyperbolic quadratic set, i.e., a Klein Quadratic set. This construction is reversible. From the nature of the coordinatization it is easy to show that the image points also lie on a quadric. This leads to the result that:

Theorem: Every Klein Quadratic Set is a quadric.