Quadratic Sets

4.6 The Klein Quadratic Set

The Klein Quadratic Set

Def:A hyperbolic quadratic set of a 5-dimensional projective space is also called a ***Klein quadratic set***.

We define a new geometry based on a Klein quadratic set.

Let C and C^* be the two equivalence classes of planes of a Klein quadratic set Q in a 5-dimensional projective space **P**. We define the geometry **S** as follows:

a) the *points* of **S** are the planes of C.

b) the *lines* of **S** are the points of Q.

c) the *planes* of **S** are the planes of C^*.

d) the *incidence* between a line of **S** and either a point or plane of **S** is induced by the incidence in **P**.

e) a point π_1 of **S** is *incident* with a plane π_2 of **S** iff the planes π_1 and π_2 of **P** are not disjoint (then by 4.5.5 they intersect each other in a line of **P**).

Counts

Lemma 4.6.1: Let $\mathcal{Q}$ be a Klein quadratic set.
a) Each $\mathcal{Q}$-line is on exactly one plane of each equivalence class.
b) If **P** is finite of order q, then each point of $\mathcal{Q}$ is on exactly q+1 planes of each equivalence class.

Pf: a) Let g be an arbitrary $\mathcal{Q}$-line. Consider a point P on g. We know that $\mathcal{Q}_P \cap \mathcal{Q}$ is a cone over a hyperbolic quadratic set $\mathcal{Q}''$.

The line g meets $\mathcal{Q}''$ in some point. Through this point there pass two $\mathcal{Q}''$-lines, h and h', one in each of the two classes of $\mathcal{Q}''$. It follows that the planes <P, h> and <P, h'> (which intersect in the line g) are the planes of the two equivalence classes through g.

b) By Theorem 4.5.4, each point of $\mathcal{Q}$ is on $(q+1)^2$ $\mathcal{Q}$-lines. Each $\mathcal{Q}$-plane through the point contains q+1 $\mathcal{Q}$-lines. Since the planes in an equivalence class meet only at the point, there will be q+1 planes of each equivalence class through the point. ❑

3-space

Theorem 4.6.2: The geometry **S** is a 3-dimensional projective space; more precisely, **S** is isomorphic to a 3-dimensional subspace of **P**.

Pf: We proceed in several steps.

Claim 1: Any two distinct points of **S** are incident with exactly one line of **S**.

This follows directly from 4.5.5 since any two planes of *C* (two points of **S**) intersect in precisely one point of *Q* (a line of **S**).

Claim 2: If two points π, π' of **S** are incident with a plane π_2 of **S** then any point of the line determined by them is also incident with π_2 (hence π_2 is a linear set). Moreover, the structure of the points and lines of **S** incident with π_2 is a projective plane.

3-space

Theorem 4.6.2: The geometry **S** is a 3-dimensional projective space; more precisely, **S** is isomorphic to a 3-dimensional subspace of **P**.

Pf(cont.):
Since π and π' are incident with π_2 in **P** the planes π and π' intersect the plane π_2 in lines g and g'. The intersection of these lines is a point P which is the common point of π and π'. The point P is the line of **S** through the points π and π' of S. Since π_2 is in *C'*, any other plane from *C* through P intersects π_2 in a line, thus in **S** the point that it represents is on the line represented by P and is incident with the plane π_2 of **S**.

By Lemma 4.6.1 each line of π_2 is on exactly one plane of *C*. Thus the structure of points and lines of **S** incident with π_2 is the structure of lines and points of **P** on π_2. This is the dual plane of π_2 and since π_2 is Desarguesian it is self-dual, thus isomorphic to π_2.

3-space

Theorem 4.6.2: The geometry **S** is a 3-dimensional projective space; more precisely, **S** is isomorphic to a 3-dimensional subspace of **P**.

Pf(cont.): Claim 3: Any three points of **S** that are not on a common line of **S** are incident with precisely one plane of **S**.

Let π_1, π_2 and π_3 be three points of **S** that are not incident with a common line of **S**. These are three planes from *C* not through a common point of *Q*. Thus, the points $P_1 = \pi_2 \cap \pi_3$, $P_2 = \pi_1 \cap \pi_3$ and $P_3 = \pi_1 \cap \pi_2$ are distinct points of *Q*. We now show that the plane π from C' through P_1 and P_2 also contains P_3. Now π intersects π_1 and π_2 in lines g_1 and g_2 since π is in the other equivalence class and has a nonempty intersection with each plane. These two lines intersect in π at a point $X = g_1 \cap g_2 = (\pi \cap \pi_1) \cap (\pi \cap \pi_2) \subseteq \pi_1 \cap \pi_2 = P_3$.

Thus, the plane π of **S** is incident with the points π_1, π_2 and π_3 of **S**. Any other plane of **S** through these points would have a line in common with π_2 and a line in common with π_3, thus it would pass through P_1. Similarly, it would pass through P_2 and P_3 and so would equal π.

3-space

Theorem 4.6.2: The geometry **S** is a 3-dimensional projective space; more precisely, **S** is isomorphic to a 3-dimensional subspace of **P**.

Pf(cont.): These three claims imply that **S** is a projective space.

Claim 4: The projective space **S** has dimension 3.

For this we show that each line of **S** and each plane of **S** are incident with a common point of **S**. It then follows that each plane of **S** is a hyperplane, so **S** has dimension 3.

Therefore, let P be a line of **S** and π_2 a plane of **S** which are not incident. So P is a point of *Q* and π_2 a plane in *C′* such that P is outside of π_2. The tangent hyperplane Q_P intersects π_2 in a line g. Then $\pi_1 = \langle P,g \rangle$ is a *Q*-plane, which is in *C* since π_1 and π_2 have a line in common. Hence, π_1 is a point of **S** that is incident with the line P and the plane π_2 of **S**. ❑