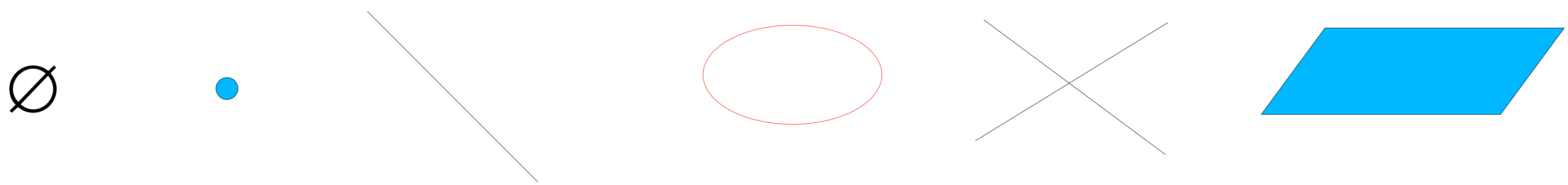
Quadratic Sets

4.3 In Spaces of Small Dimension
4.4 In Finite Projective Spaces

Quadratic Sets in a Plane

Def: A nonempty set O of points in a projective plane is called an ***oval*** if no three points of O are collinear and each point of O is on exactly one tangent.

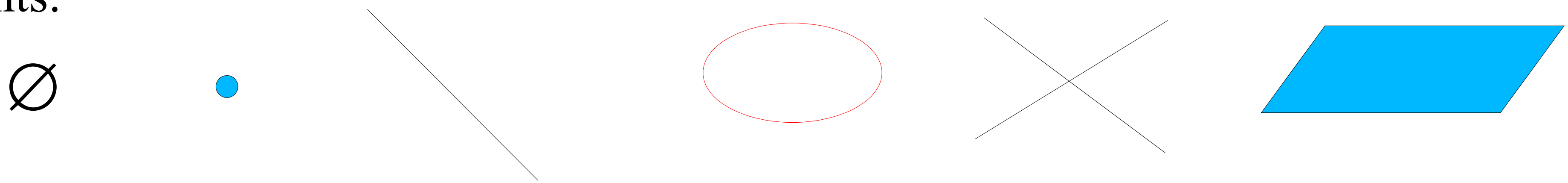
Theorem 4.3.1: Let Q be a quadratic set in a projective plane **P**. Then Q is the empty set, just one point, one line, an oval, the set of points on two lines, or the whole set of points.



Hence there is only one type of nonempty, nondegenerate quadratic sets in a projective plane, namely the ovals.

Quadratic Sets in a Plane

Theorem 4.3.1: Let *Q* be a quadratic set in a projective plane **P**. Then *Q* is the empty set, just one point, one line, an oval, the set of points on two lines, or the whole set of points.



Pf: Each of these sets is a quadratic set, so we need to prove that there are no others. We may assume that *Q* is not a subspace, so it contains at least two points.

First, assume that *Q* contains no line. It follows that no three points of *Q* are collinear. Since *Q* contains at least 2 points, at any point P of *Q*, there passes a line which is not a tangent, so Q_P is not the whole space and *Q* must be nondegenerate. In other words, Q_P is a hyperplane, which in this case is a line, i.e., through each point of *Q* there passes a unique tangent line. Hence, *Q* is an oval.

Quadratic Sets in a Plane

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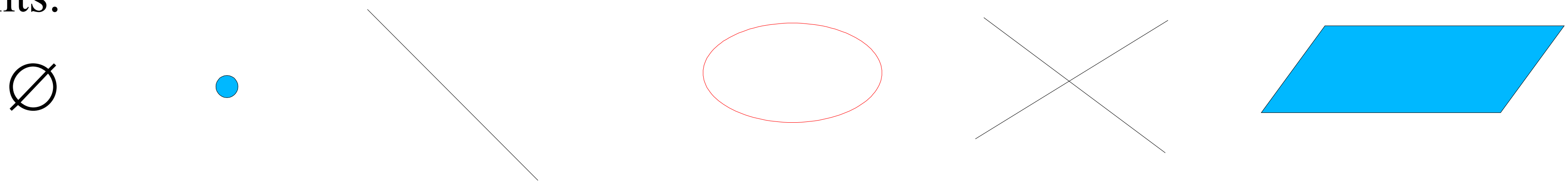


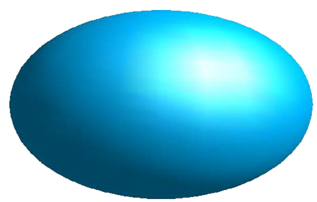
Pf(cont.): Now assume that *Q* contains a line (i.e., *Q* has index at least two). If l is a line contained in *Q*, since *Q* is not a subspace there must be a point R of *Q* not on l. Any tangent line through R (and there must be at least one) meets l, and so must be contained in *Q*. Thus, *Q* contains at least two lines and each point of *Q* is on a *Q*-line. Let P be the intersection of these two *Q*-lines. All the lines through P are tangent lines, so Q_P = P. Suppose there is a third *Q*-line. If this line does not pass through P, then all the lines through P are *Q*-lines and *Q* = **P**. If this line passes through P, then any line not through P will have three points of *Q* and so is a *Q*-line, and again *Q* = **P**. ❑

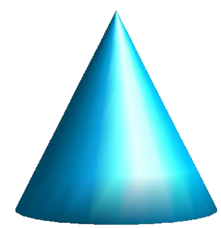
Ovoids, Cones and Hyperboloids

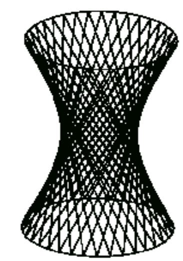
Def: Let **P** be a d-dimensional projective space. An ***ovoid*** is a nonempty set *O* of points of **P** satisfying :

1. No three points of *O* are collinear.
2. For each point P on *O*, the tangents through P cover exactly a hyperplane.

Now let d = 3.







A set *K* of points of **P** is called a ***cone*** if there are a plane π, and oval *O* in π, and a point $V \notin \pi$ such that *K* consists of the points on the lines VX with $X \in O$. We call V the ***vertex*** of the cone *K*.

A ***hyperboloid*** is the set of points incident with the lines of a regulus.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).

In particular, the nonempty, nondegenerate quadratic sets in a 3-dimensional projective space are precisely the ovoids and the hyperboloids.

Pf: Suppose that Q is not a subspace, so Q contains at least two points.

If Q has index 1 then no three points of Q are collinear, in particular Q is nondegenerate. Q is therefore an ovoid.

If Q has index 3, the maximum possible since Q is not the whole space, then analogous to the 2-dimensional argument, Q must be the union of two planes.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).
In particular, the nonempty, nondegenerate quadratic sets in a 3-dimensional projective space are precisely the ovoids and the hyperboloids.

Pf(cont.): We now consider the case that Q has index 2 (the maximum possible for nondegenerate quadratic sets by Theorem 4.2.4). We deal with this case in several steps.

Step 1: $\dim(\mathrm{rad}(Q)) \leq 0$.

Assume that rad(Q) contains a line g. Since $Q \neq g$ there is a point P of Q not on g. This implies $\langle P, g\rangle \subseteq Q$, contradicting the fact that Q has index 2.

Step 2: The number of Q-lines through a point of Q outside of the radical is 1 or 2.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).

Pf(cont.): By Lemma 4.2.1 each point P of Q is on at least one Q-line. Assume that P is on three Q-lines. If these lines are not in a common plane then $Q_P = P$, hence $P \in \mathrm{rad}(Q)$, a contradiction. Hence, the three lines lie in a common plane π. Since $Q \cap \pi$ is a quadratic set, but in a plane the only quadratic set that contains 3 lines is the whole plane, so $\pi \subseteq Q$, contradicting the fact that the index of Q is 2.

Step 3: Each point of Q not in rad(Q) lies on the same number of Q-lines.

Assume that there are two points P_1, P_2 in Q such that P_1 is on one Q-line g_1 and P_2 is on two Q-lines, g_2,g_3.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).

Pf(cont.): The tangent plane at P_1 contains g_1 and the only points of Q in this plane are those on g_1. The tangent plane at P_2 is spanned by g_2 and g_3 and the points of Q in it are on these lines. These two tangent planes are therefore distinct and so meet in a line g.

In the tangent plane at P_1, the line g either meets g_1 in a point, or is equal to g_1. In the first case the line g has exactly one point of Q on it. In the tangent plane at P_2, it is clear that the line g must pass through P_2. Thus, P_2 lies on g_1 in this case. If on the other hand $g = g_1$, and must also contain P_2. For any point P on g_1, any line in the tangent plane at P_1 through P is a tangent line, so is contained in Q_P. Thus, the tangent plane at P_2 contains (and hence is equal to, since P_2 is not in the radical) the tangent plane at P_1, a contradiction.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).

Pf(cont.):

Step 4: If each point of $Q\backslash\text{rad}(Q)$ is on just one Q-line then Q is a cone. If g and g' are two Q-lines then since the rad(Q) contains at most one point, there is a point P not in rad(Q) on g. By Lemma 4.2.1, the point P must be joined with g' by a Q-line. Since g is the only Q-line on P, g must intersect g'. Therefore, any two Q-lines meet.

By the hypothesis, any two Q-lines must intersect in rad(Q). So rad(Q) consists of a point V and all Q-lines pass through V.

Let π be a complement of V (which must be a plane). By Lemma 4.1.2 Q induces a nondegenerate quadratic set Q' in π. Since Q has index 2, $Q \neq Q'$. Moreover, Q' has index 1, so it must be an oval. Thus Q is a cone.

3-dimensional Quadratic Sets

Theorem 4.3.2: Let Q be a quadratic set in a 3-dimensional projective space **P**. Then Q is a subspace, an ovoid, a cone, a hyperboloid or the union of two planes (hyperplanes).

Pf(cont.):

Step 5: If each point of $Q \backslash \text{rad}(Q)$ is on two Q-lines then Q is a hyperboloid.

Let g be a Q-line. Since no three Q-lines are in a common plane, all Q-lines that meet g are skew. Denote the set of these lines by R. Since any point of Q is joined to g by a Q-line, the lines in R cover all the points of Q.

Suppose X in rad(Q). We can find a Q-line that does not contain X (since the lines of R are skew), so we may assume that X is not on g. All the lines through X in <X,g> are tangents contained in Q, so Q would have index 3, a contradiction. Therefore, Q is nondegenerate.

Through any point P on a line h of R there is a unique Q-line h' ≠ h. Since P is connected to any line of $R \backslash \{h\}$ by a Q-line, the line h' meets every line of R. Hence R is a regulus, and Q is a hyperboloid. ❑

Counts in Finite Projective Spaces

In this section let **P** = PG(d,q) and let Q be a quadratic set in **P**.

Lemma 4.4.1: For a point P$\in Q\backslash$rad(Q) we denote by a (= a_P) the number of Q-lines through P. Then:

(a) If Q_P is a hyperplane then Q_P contains exactly aq + 1 points of Q.

(b) We have $|Q| = 1 + q^{d-1} + a$q; in particular, a is independent of the choice of the point P$\in Q\backslash$rad(Q).

Pf: Each line through P in Q_P contains either no or q further points of Q. The lines through P not in Q_P contain exactly one further point of Q.

On the a Q-lines through P in $Q\backslash$rad(Q) there are exactly 1 + aq points of Q. Since these are all the point of Q in Q_P, we have proved (a). All the lines through P not in Q_P intersect Q in a second point. Since there are exactly q^{d-1} such lines, (b) also follows. ❑

Examples

1. In the plane (d = 2), an oval contains no lines ($a = 0$), so the number of points on an **oval** is $1 + q$. For the **pair of intersecting lines**, a point not equal to the point of intersection, is on exactly one line contained in the set ($a = 1$), so the number of points is $1 + q + q = 2q + 1$.

2. In 3-space (d = 3) an **ovoid** contains no lines ($a = 0$) and so has $1 + q^2$ points; there are 2 lines through each point of a **hyperboloid** ($a = 2$) in the hyperboloid and so there are $1 + q^2 + 2q = (q+1)^2$ points; and through each point other than the vertex of a **cone** there passes one line of the cone ($a = 1$), so the cone contains $1 + q^2 + q$ points.

4-dimensional Quadratics

Theorem 4.4.2: Any nonempty, nondegenerate quadratic set in **P** = PG(4,q) has index 2.

Pf: Let Q be a nonempty, nondegenerate quadratic set in **P**. By Theorem 4.2.4 we need only show that Q has index at least 2.

Assume that Q has index 1. Then $a = 0$ and, by Lemma 4.4.1(b), Q has $q^3 + 1$ points.

If **H** is a hyperplane of **P** that contains at least two points of Q, then the induced set $Q' = Q \cap \mathbf{H}$ is an ovoid. This follows since Q' is a quadratic set of index 1 and is non-degenerate since since it contains at least 2 points. In particular, Q' has exactly $q^2 + 1$ points.

We shall now compute the number of these hyperplanes. Through a fixed point P of Q there is a tangent hyperplane Q_P and all other hyperplanes must contain at least one other point of Q. Thus there are $q^3 + q^2 + q$ of these hyperplanes through a point of Q.

4-dimensional Quadratics

Theorem 4.4.2: Any nonempty, nondegenerate quadratic set in **P** = PG(4,q) has index 2.

Pf(cont.): The number of these hyperplanes is thus

$$\frac{|Q|(q^3+q^2+q)}{q^2+1}$$

Since each point of *Q* is counted $q^2 + 1$ times.

As this must be an integer, q^2+1 must divide $(q^3+1)(q^3+q^2+q)$, but the remainder of this division is $q + 1$ which can not be 0. ❑

Even dimensional Projective Spaces

Theorem 4.4.3: Any nonempty, nondegenerate quadratic set in **P** = PG(2t,q) has index t.

Pf: Let Q be a nonempty, nondegenerate quadratic set of **P**. By Theorem 4.2.4 we only have to show that the index of Q is at least t. We proceed by induction on t.

The case t = 1 was treated in Theorem 4.3.1, and the case t = 2 in Theorem 4.4.2, so we may assume that t > 2 and assume that the assertion is true for t-1.

We first claim that the index of Q is at least 2. Assume that the index of Q is 1. Each tangent hyperplane would thus contain just one point of Q. Let P and P' be distinct points of Q and let **U** be a (2t-2)-dimensional subspace of **P** which contains both of them. The quadratic set $Q' = Q \cap \mathbf{U}$ is non-empty. Consider the tangent hyperplane Q'_R at a point R of Q'.

Even dimensional Projective Spaces

Theorem 4.4.3: Any nonempty, nondegenerate quadratic set in **P** = PG(2t,q) has index t.

Pf(cont.): By Lemma 4.1.1 we have $Q'_R = Q_R \cap \mathbf{U}$. Since Q_R contains only one point of **U**, it follows that $Q_R \neq \mathbf{U}$. Therefore, Q' is a non-empty, nondegenerate quadratic set of index 1 in the 2(t-1)-dimensional projective space **U** with t > 1, a contradiction.

Thus the index of Q is at least 2. Now consider a point P of Q and its tangent hyperplane $\mathbf{H} = Q_P$. Let **W** be a complement of P in **H**, that is, a subspace of dimension 2t-2 of **H** that does not contain P. Let $Q' = Q \cap \mathbf{W}$ be the quadratic set induced by Q in **W**. Since the index of Q is at least 2, there is at least on Q-line through P, in particular, Q' is not empty. By Theorem 4.1.4, Q' is non-degenerate.

Thus, by induction the index of Q' equals t-1. So Q' contains a Q-subspace **U** of dimension t-2. It follows that <P,**U**> has dimension t-1 and is contained in Q. Thus Q has index at least t. ❑

Witt's Theorem

Theorem 4.4.4: (**Witt's Theorem**) The index s of a nonempty, nondegenerate quadratic set in **P** = PG(d,q) is either d/2 if d is even, or s = (d-1)/2 or (d+1)/2 if d is odd.

Pf: The case of d even was dealt with in Theorem 4.4.3. Thus, suppose that d = 2t + 1. By Theorem 4.2.4 the index of *Q* is at most t+1.

We must now show that s ≥ t. Consider a hyperplane **H** that contains at least one point of *Q* and is not a tangent hyperplane. Such a hyperplane exists. Counting the point – hyperplane flags (P,**W**) with P in *Q*, we get on the one hand $|Q|(q^{d-1} + ... + 1)$ since there are $q^{d-1} + ...+ 1$ hyperplanes through a point. On the other hand there are |*Q*| tangent hyperplanes, each containing 1 + *a*q points of *Q*. Since *Q* is nondegenerate not all points of a tangent hyperplane lie in *Q*, therefore $a < q^{d-2} +... + 1$. Thus there must be a hyperplane through any point of *Q* which is not a tangent hyperplane.

Witt's Theorem

Theorem 4.4.4: (**Witt's Theorem**) The index s of a nonempty, nondegenerate quadratic set in **P** = PG(d,q) is either d/2 if d is even, or s = (d-1)/2 or (d+1)/2 if d is odd.

Pf(cont.):The index s' of the quadratic set $Q' = Q \cap \mathbf{H}$ satisfies s' ≤ s. Moreover, by Theorem 4.1.4 Q' is nondegenerate and by construction Q' is not empty.

Theorem 4.4.3 yields s' = t, and therefore s ≥ s' = t. ❑