

# The Representation Theorems

The Representation of Projective Spaces

# The Representation Theorem

**Theorem 3.4.2:** Let  $\mathbf{P}$  be a projective space of dimension at least 2. If  $\mathbf{P}$  is Desarguesian, then there is a vector space  $V$  over a division ring  $F$  such that  $\mathbf{P}$  is isomorphic to  $\mathbf{P}(V)$ . In particular, every projective space of dimension at least 3 is isomorphic to a  $\mathbf{P}(V)$ .

An immediate corollary of this result is:

**Theorem 3.4.1:** Let  $\mathbf{A} = \mathbf{P} \setminus \mathbf{H}$  be an affine space. If the theorem of Desargues is valid in  $\mathbf{A}$  then there exist a division ring  $F$  and a vector space  $V^*$  over  $F$  such that

- a. the points of  $\mathbf{A}$  are the elements of  $V^*$ .
- b. the lines of  $\mathbf{A}$  are the cosets of the 1-dimensional vector subspaces of  $V^*$ .

# The Representation Theorem

We will not give the details of the proofs of these results, but only outline the structure of the proofs.

In order to prove the result, we must find an appropriate vector space, which amounts to finding a division ring, since the vector space can then be constructed having the appropriate dimension. The division ring is constructed from the central collineations of the given projective space. The Desarguesian property is needed to ensure that all the collineations we need in the construction actually exist.

# The Translation Group

The set of our division ring is essentially the set of all central collineations of the projective space  $\mathbf{P}$  with a fixed axis  $\mathbf{H}$  and fixed center  $O$  (not in  $\mathbf{H}$ ). Multiplication on this set will be defined as composition of the maps - thus we automatically have a group under multiplication which is generally non-abelian having the identity map as the multiplicative identity. Addition is a little trickier since it must be a commutative operation. To help define addition, we consider the group of all elations with axis  $\mathbf{H}$  under composition. In the affine space  $\mathbf{P} \setminus \mathbf{H}$  these maps are called *translations*, so this group is usually referred to as the *translation group*  $T(\mathbf{H})$ . This group is abelian and has the property that for any two affine points  $P$  and  $Q$ , there is a *unique* element of the group which maps  $P$  to  $Q$ .

# Adding Points

We use this group to define an addition on the set of affine points of  $\mathbf{P}$  (as an intermediate step towards getting a definition of addition of central collineations).

For each point  $P$  of  $\mathbf{P}\setminus\mathbf{H}$ , there is a unique element of  $T(\mathbf{H})$  which maps  $O$  to  $P$ . We can therefore identify the points of  $\mathbf{P}\setminus\mathbf{H}$  with these maps, the map corresponding to point  $P$  will be denoted by  $t_P$ . Addition of points  $P$  and  $Q$  is then defined as

$$P + Q = t_P(t_Q(O)) = t_P(Q).$$

Under this addition, the set of affine points form an abelian group which is clearly isomorphic to  $T(\mathbf{H})$ .

# Adding Collineations

We are now ready to define addition of our central collineations. If  $s$  and  $r$  are two collineations from our set we define

$$(s + r)(X) = s(X) + r(X)$$

where the addition on the right is the addition of points we have just defined. We now need to show that this defines a central collineation in our set. It turns out that it almost does, the one exception is that you may get the function  $(s + r)(X) = O$  for all  $X$ . We call this function the *zero map* and it is not a central collineation. But, if we add the zero map to our set of collineations, addition will be a closed operation and the zero map will be the additive identity. We can see that the composition of any central collineation with the zero map will be the zero map. This addition is abelian and the only thing that needs to be checked is that the distributive laws hold to show that we have a division ring.

# The Conclusion

Now that we have a division ring, call it  $F$ , we can construct a vector space  $V$  over  $F$  of dimension  $d + 1$  where  $\mathbf{P}$  has dimension  $d$ . We then need to define a map from  $\mathbf{P}$  to  $\mathbf{P}(V)$  and show that it is an isomorphism. To simplify the definition of this map, we construct the vector space  $V$  as the set of ordered pairs  $(b, P)$  where  $b \in F$  and  $P$  is an affine point of  $\mathbf{P} \setminus \mathbf{H}$ . We then define the map  $\alpha$  from  $\mathbf{P}$  to  $\mathbf{P}(V)$  by:

$$\begin{aligned} \alpha(X) &= \langle (1, X) \rangle \quad \text{when } X \in \mathbf{P} \setminus \mathbf{H}, \text{ and} \\ \alpha(X) &= \langle (0, Y) \rangle, \quad Y \text{ on } OX, Y \neq O \text{ or } X, \text{ when } X \in \mathbf{H}. \end{aligned}$$

This point map needs to be shown to be well-defined (because of the second part of the definition), and then that it truly is an isomorphism.