

Regular Two-graphs (cont.)

There is another characterization of regular two-graphs. Let Γ_x denote the unique graph in a given switching class (two-graph) for which x is an isolated vertex (that is, for which the two-graph design is the extension obtained by adjoining x). For the regular two-graph design of (4.10), Γ_x is the pentagon (for any vertex x).

4.11. Theorem. *For a non-trivial two-graph design (X, \mathcal{B}) , the following are equivalent:*

- (a) (X, \mathcal{B}) is regular;
- (b) for all $x \in X$, Γ_x is strongly regular with $k = 2\mu$;
- (c) for some $x \in X$, Γ_x is strongly regular with $k = 2\mu$.

Proof. Let k_y be the degree of y , and γ_{yz} the number of common neighbors of y and z in the graph Γ_x . Then the number of triples containing x and y is k_y , and the number containing y and z is $k_y + k_z - 2\gamma_{yz}$ if y and z are non-adjacent, or $n - k_y - k_z + 2\gamma_{yz}$ if y and z are adjacent. So, (X, \mathcal{B}) is regular if and only if Γ_x is strongly regular with parameters satisfying

$$k = 2(k - \mu) = n - 2(k - \lambda).$$

Now the equation

$$k(k - \lambda - 1) = (n - 2 - k)\mu$$

shows that, if $k = 2\mu$, then $n - 2(k - \lambda) = k$; so the second equality is a consequence of the first. \square

4.12. Corollary. *A non-trivial regular two-graph design has an even number of points.*

Proof. $n = 3k - 2\lambda$, and $k = 2\mu$ is even. \square

We have another link between strongly regular graphs and regular two-graphs.

4.13. Theorem. *Let Γ be a regular graph which yields the two-graph design (X, \mathcal{B}) . Then (X, \mathcal{B}) is a regular two-graph design if and only if Γ is strongly regular with eigenvalues k, r, s satisfying $n = 2(k - r)$ or $n = 2(k - s)$.*

Proof. If A is the usual adjacency matrix of Γ , then the Seidel matrix B is given by $B = J - I - 2A$. Because Γ is regular, A commutes with J , and so A and B have the same eigenvectors.

If the two-graph is strongly regular, then B has only two distinct eigenvalues, and so $A|_{\mathbf{j}^\perp}$ has only two distinct eigenvalues; so Γ is strongly regular by (2.19).

Conversely, if Γ is strongly regular, then \mathbf{j} is an eigenvector of B with eigenvalue $n - 1 - 2k$, and the other eigenvalues of B are $-1 - 2r$ and $-1 - 2s$. So by (4.9), (X, \mathcal{B}) is regular if and only if either $n - 1 - 2k = -1 - 2r$ or $n - 1 - 2k = -1 - 2s$. \square

We can now finish the classification of strongly regular graphs with least eigenvalue -2 .

4.14. Theorem. *A strongly regular graph with least eigenvalue -2 is one of the following:*

- (a) $T(m)$, $m \geq 5$;
- (b) $L_2(m)$, $m \geq 3$;
- (c) $CP(m)$, $m \geq 2$;
- (d) the Petersen graph $(10,3,0,1)$;
- (e) the complement of the Clebsch graph $(16,10,6,6)$;
- (f) the complement of the Schläfli graph $(27,16,10,8)$;
- (g) the Shrikhande graph $(16,6,2,2)$;
- (h) the three Chang graphs $(28,12,6,4)$.

Proof. By (3.10), either we have (a)-(c), or our graph Γ is represented in E_8 . In the latter case, $A+2I$ has rank at most 8, and so the eigenvalue r of A has multiplicity at most 7. By (2.23), Γ has at most 35 vertices. Now it is straightforward to list all the possible parameters satisfying the integrality and Krein conditions and in which $s = -2$ and r has multiplicity 5, 6 or 7. The resulting list consists of the parameters of (d) - (h) in the theorem and of $T(6)$, i.e., $(15,8,4,4)$.

For the parameter sets $(15,8,4,4)$ and $(27,16,10,8)$, the complementary graph has parameters $(15,6,1,3)$ and $(27,10,1,5)$ respectively. By (2.5) the original graphs are unique and are $T(6)$ and the complement of the Schläfli graph respectively.

In the remaining cases, with 10, 16 or 28 vertices, it happens that all possible parameter sets satisfy the conditions of (4.13), and so the unknown graph yields a regular two-graph (X, \mathcal{B}) . By (4.11), the derived graph, Γ_x , is strongly regular with parameters $(9,4,1,2)$, $(15,6,1,3)$ or $(27,10,1,5)$ respectively. Again, by (2.5), we know all these graphs! We conclude that, for each of the three numbers of vertices, the unknown strongly regular graphs fall into a single switching class, and thus are obtained from known ones by switching. We have to investigate how this switching can be done. We give the argument for $n = 28$; the other cases are considerably simpler.

Our problem, then is to find all switching sets Y of vertices which switch $T(8)$ into a regular graph with degree 12. As in (4.7), the switching set is the edge set of a graph Δ on 8 vertices, and a short argument shows that this graph is regular. Moreover, we can assume that Y is smaller than its complement, from which it follows that Δ has degree at most 3.

Before proceeding further, we elaborate on (4.7), where we observed that the graph obtained by switching $T(8)$ with respect to the edge set of $2K_4$ (the disjoint union of two K_4 s) is isomorphic to $T(8)$. If the vertex sets of the K_4 s are $\{1,2,3,4\}$ and $\{5,6,7,8\}$, then an isomorphism between the unswitched and switched graphs is given by the permutation which interchanges the pair $\{i,j\}$ that is contained in one of the 4-sets with its complement in that 4-set and otherwise fixes a 2-set. We call this process *reflection*.

If Δ and Δ' are regular graphs on $\{1,\dots,8\}$, and if Δ' is obtained from Δ by taking the symmetric difference of its edge set with $2K_4$ followed by reflection, then the graphs obtained by switching $T(8)$ with respect to Δ and Δ' are isomorphic. We call this the *SDR* (=“symmetric difference plus reflection”) *process*.

In particular, $2K_4$ is equivalent to the null graph under SDR. Also, $2C_4$ (two disjoint 4-cycles) is equivalent to $4K_2$ (four disjoint edges). In the figure below we

show a less trivial instance of SDR, where the complete graphs have vertex sets $\{1,2,5,8\}$ and $\{3,4,6,7\}$ respectively. The first graph is the original graph G . The second is the result of applying symmetric difference with respect to these two K_4 s (note that edges between the complete graphs are unchanged). The third graph is obtained by reflection of the second graph and so is an SDR of G .

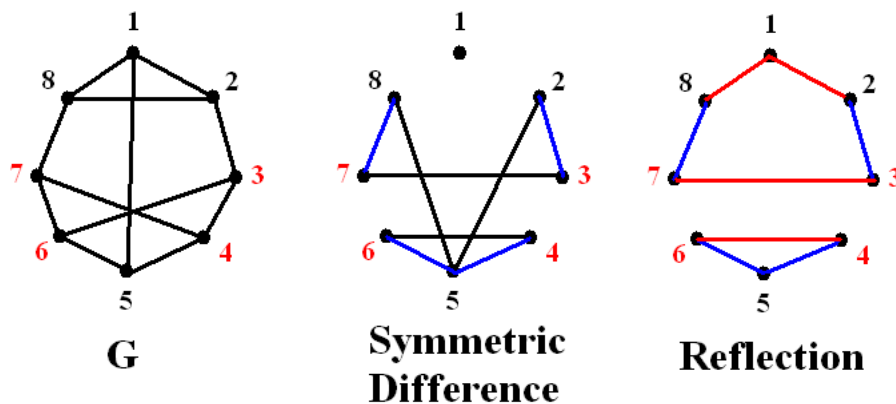


FIGURE 1. Symmetric Difference and Reflection

Now we examine regular graphs of degree m on 8 vertices, for $m = 1, 2, 3$.

Case $m = 1$: The only graph of degree 1 is $4K_2$, which gives rise to Chang(1).

Case $m = 2$: The three graphs are C_8 , $C_5 + C_3$, and $2C_4$. The first two give Chang(2) and Chang(3); the third gives Chang(1) again, by SDR.

Case $m = 3$: There are six graphs of degree 3 on 8 vertices. Use of SDR shows that no new strongly regular graphs arise.

Finally, note that the Chang graphs are not isomorphic to $T(8)$ (since none contains a 7-clique), nor to one another (they contain different numbers of 6-cliques, 8, 0 and 3 respectively).

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