

Georg Friedrich Bernhard Riemann

Insight \implies Rigor
A Biography of an Intuitive Mathematician

by Edward Gard
Math 4010 Spring 2008

Edward Gard
Georg Friedrich Bernhard Riemann
Math 4010 – Project Biography
Professor Cherowitzo
Spring 2008

Abstract

The Riemann Hypothesis is the greatest unsolved problem in mathematics. Mathematicians have worked on it for over 150 years, with no success. This paper looks at the life of Bernhard Riemann and contrasts one of his greatest strengths – Mathematical Insight – with its opposing force, mathematical rigor.

Contents

1	Introduction	4
2	A Brief Life	5
3	Collected Works	7
4	A Quiet Revolution	8
5	Riemann's Insight	8
6	Avoid Riemann!	10
7	Conclusion –Between a Rock and a Hard Place	13

List of Figures

1	Memorial tablet in Breselenz	6
2	Zeros Fall On Line That Extends Upwards	9

Edward Gard
Georg Friedrich Bernhard Riemann
Math 4010 – Project Biography
Professor Cherowitzo
Spring 2008

“If I only had the theorems, then I could find the proofs easily enough.”

Riemann

“I have had the results for a long time, but I do not know yet how to arrive at them”

Gauss

1 Introduction

In the mathematical landscape, a glimpse of the summit comes in a flash. It is the summit that all aspire to conquer, but without the summit, there is no need for a journey. With the summit, the struggle begins with careful planning and practice. Only a fool would attempt the summit without proper preparation.

A natural tension lies between the summit’s discovery and the journey to it. Call it the struggle between insight and rigor. In mathematics, the summit is the theorem or conjecture and the journey to it is the theorem’s rigorous proof. On the journey to the summit, the slightest misstep results in dire consequences. Many have lost limbs or even their lives due to the lack of rigor. It is only natural to put the greatest emphasis on rigor. There are times, however, when the destination is paramount. Who wants to waste there precious time proving a useless theorem?

Clearly, insight and rigor are forever connected. Put boldly forth, rigor follows insight and the stage is set to look at the Georg Bernhard Riemann's life, always keeping in mind that insight implies (\Rightarrow) rigor.

2 A Brief Life

Using *The life of Bernhard Riemann* by R. Dedkind¹ in Bernhard Riemann's *Collected Papers* (see [7]) and Detlef Laugwitz's work (see [5]) a brief history of Riemann's life may be constructed as follows:

Georg Friedrich Bernhard Riemann was born on September 17, 1826 in Breselenz, in the Kingdom of Hanover. He was the second of six children born to his father, Friedrich Bernhard Riemann who was a pastor in Breselenz. Breselenz is a small village in Hanover.

The father, Friedrich, was born in Boitzenburg, not far from Breselenz. He was a lieutenant under Wallmoden in the war of liberation and married Charlotte, the daughter of Privy Councillor Ebell of Hanover. It was his father who taught his young son until the son left to go to the Gymnasium, which is similar to a U.S. college preparatory high school. At the age of ten Riemann's father hired a tutor, Schulz, to assist young Riemann's education. Schulz provided arithmetic and geometry to young Riemann who soon excelled with better and faster answers than his teacher. Riemann entered the University of Göttingen and his father naturally hoped young Riemann would follow in his footsteps and devote himself to the

¹This essay developed by Dedekind's correspondence.

study of theology, but it was mathematics where young Riemann's talent lay and his father quickly added his support.

Georg Friedrich Bernhard Riemann attended the following Gymnasiums: Hanover(1840-1842) and Lüneburg(1842-1846). He went on to Göttingen (1846-1847, 1849-1851) and Berlin(1847-1849) and earned his doctorate in (1851). Riemann Habiliated (qualifying exam to lecture at a university) in 1854 at Göttingen under Gauss, became an Associate professor at Göttingen in 1857 and a Full professor in 1859. Riemann married Elise Koch, friend of his sister in 1862 and spent many years in Italy. Elise and Georg had only one daughter, Ida, who was born in Italy. Most of his life Riemann was in poor health and he died on July 20, 1866 in Selasca on Lago Maggiore. His Memorial tablet is found in Breselenz, a photo in [5, p13] reads as follows (see Figure 1):

Georg Friedrich
Bernhard
Riemann
*17.9.1826 Breselenz
20.7.1866
Selasca-Lago Maggiore

Mathematiker
Professor in Göttingen

Gewidmet
Zum 150. Geburtstag

Figure 1: Memorial tablet in Breselenz

Riemann would have been forty on September 17, 1866 had he lived two more months and one can only speculate what other contributions he may have accom-

plished, if his health was better. His collected works list thirty-one papers (see [7]). These are discussed in the next section.

3 Collected Works

From Riemann's famous hypothesis, one would conclude that his works would contain many papers on number theory, but this is not the case. Only the one work deals with number theory. There are, however, a richness and variation in his collection (see [7]).

He did work in complex variables, distribution of electric charges, Nobili color rings, theory of functions, Abelian functions, primes less than a given magnitude (*The Riemann Hypothesis*), propagation of planar air waves, motion of homogeneous fluid, θ -functions, trigonometric series, geometry, electrodynamics, periodic functions, surface of least area, attempted generalization of integration and differentiation, residual charge, linear differential equations, hypergeometric series, ring potential, diffusion of heat, equilibrium of electricity and elliptic modular functions.

Besides his hypothesis, the *Riemann Integral* is also well known. For a quick introduction, see [10].

With such a large body of work, what type of environment did Riemann work in? This is the purpose of the next section.

4 A Quiet Revolution

Devlin writes in [2] about the quiet mathematical revolution that occurred in the middle of the nineteenth century. At the University of Göttingen the leaders of the revolution were as follows: Lejeune Dirichlet, Richard Dedekind and Riemann. Mathematics moved from *doing to understanding*. Klein writes in [4] that Riemann was strongly bound to Dirichlet. It was Dirichlet who loved the intuitive substrate and would create foundational questions which avoided long computations.

Through his relationship with Dirichlet, Riemann developed his insight and clearly valued it. Devlin adds in [2] that the Göttingen revolutionaries focused on "thinking in concepts" and that the mathematical objects were no longer just a collection of formulas. These objects had conceptual properties that proved that math was not a process of transforming terms by rules, but a process of deduction from concepts. And this *deduction from concepts* is what the Göttingen revolution brought to the mathematics world.

Next is Riemann's main insight. What was it? How did it impact mathematics?

5 Riemann's Insight

Titchmarsh in [9] writes that the *Riemann Hypothesis* is the most difficult problem that the human mind has ever conceived and that Riemann's intuition is at its most powerful and mysterious.

So, briefly, what is the *Riemann Hypothesis*? Simply stated:

All the zeros of the zeta function have real part $1/2$.

First, here is the zeta function:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \dots$$

Now, the *real part* $1/2$ refers to a complex number. A complex number is in the form $a + bi$ where a and b are real numbers and $i^2 = -1$. Complex numbers are graphically represented by a coordinate system on a two dimensional plane. The horizontal axis represents the a coordinate and the vertical axis represents the i coordinate. The line at $(.5, bi)$, which extends upwards without bounds, shows where the zeros of the zeta function must fall by Riemann's hypothesis (see Figure 2). An input to a function, such as $(.5, bi)$, that results in a zero value, is called a zero of the function.

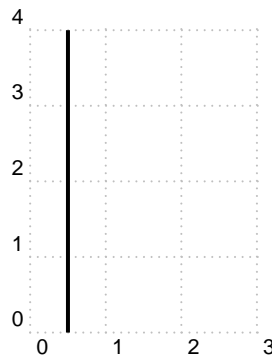


Figure 2: Zeros Fall On Line That Extends Upwards

To add precision and concreteness to this discussion, the following is an actual zero of the ζ -function, accurate to over 1000 decimal places, taken from Andrew

Odlyzko website (see [6]):

.5,

14.134725141734693790457251983562470270784257115699243175685567460149
9634298092567649490103931715610127792029715487974367661426914698822545
8250536323944713778041338123720597054962195586586020055556672583601077
3700205410982661507542780517442591306254481978651072304938725629738321
5774203952157256748093321400349904680343462673144209203773854871413783
1735639699536542811307968053149168852906782082298049264338666734623320
0787587617920056048680543568014444246510655975686659032286865105448594
4432062407272703209427452221304874872092412385141835146054279015244783
3835425453344004487936806761697300819000731393854983736215013045167269
6838920039176285123212854220523969133425832275335164060169763527563758
9695376749203361272092599917304270756830879511844534891800863008264831
2516911271068291052375961797743181517071354531677549515382893784903647
4709727019948485532209253574357909226125247736595518016975233461213977
3160053541259267474557258778014726098308089786007125320875093959979666
60675378381214891908864977277554420656532052405i

Riemann's Hypothesis will, again, be revisited in the next section at deeper level, along with more discussion of insight and rigor.

6 Avoid Riemann!

In the preface of his book (see [3]), *Riemann's Zeta Function*, Edwards writes:

On the contrary, the mathematics of previous generations is generally considered to be unrigorous and naïve, stated in obscure terms which can be vastly simplified by modern terminology, and full of false starts and misstatements which a student would be best advised to avoid. **Riemann in particular**² is avoided because of his reputation for lack of rigor, because of his difficult style, and because of a general impression that the valuable parts of his work have all been gleaned and incorporated into subsequent more rigorous and more readable works.

It looks like Edwards is a strong supporter of rigor and his book is an excellent resource for anyone interested in studying Riemann's Hypothesis in more depth and this is now a perfect place to study Riemann's original writing that generated so much mathematics to date and will continue to generate new mathematics far into the future. In *On The Number Of Prime Numbers Less Than A Given Quantity* (see english translation by Michael Ansaldi in [8, p1045], *God Created the Integers* Riemann writes:

The number of root of $\zeta(t) = 0$ whose real part lies between 0 and T is around

$$= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi};$$

for the integral $\int d \log \zeta(t)$, positively extended around the peak of

²Bold added by this paper's author

the t values (the imaginary part of which lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$, and the real part between 0 and T is (up to a fraction of the order of magnitude of $\frac{1}{T}$ equal to $(T \log \frac{T}{2\pi} - T)i$; but this integral is equal to the number of roots of $\zeta(t) = 0$ lying within this domain, multiplied by $2\pi i$. Now one does in fact find about that many real roots within these limits, and it is quite likely that all the roots are real. Certainly a rigorous proof of this would be desirable; however, after several fleeting attempts to no avail, I have temporarily set aside the search for this proof, since for the immediate purpose of my investigation it appeared to be dispensable

So, in this flash of insight, Riemann lets loose his famous hypothesis on the world. On 3/31/08, using the search term, *Riemann Hypothesis*, a google search generated 307,000 hits. And one concludes that "Yes!" insight does in fact imply rigor. Edwards work [3] alone is 315 pages in length.

This connection between insight and rigor also appeared on March 18, 2008 in Cherowitzo's Math 4010 *History of Mathematics* course, see [1]. David Hilbert (1862-1943) submitted a paper proving the finite basis theorem to the *Mathematische Annalen* and Klein who refereed the paper strongly objected to Hilbert's thoughts presented as only formal rules and concluded that the work was insufficient for the *Annalen*. This is another battle between insight, on Klein's part, and rigor, on Hilbert's part. It was at this point in the lecture that Cherowitzo explained:

If you continue to follow the formal rules back and back at some point you must have that which generated the rules. What is that?³

The paper's journey is near its end and it only requires some final concluding thoughts, which follow in the next section.

7 Conclusion –Between a Rock and a Hard Place

This paper started with a mountaineering story and looking back, a story from Greek mythology seems a fitting close (Mathematicians love symmetry, do they not?). Perhaps the sailor's peril between Scylla and Charybdis is closely related to a mathematician's tension between insight and rigor.

Scylla is a six-headed monster who dwells in the rock and enjoys eating sailors, while Charybdis creates whirlpools with its huge mouth. Only the best sailors navigate the narrow passage between the two monsters; all others perish at one of the monster's hands.

So, only the best survive. Only the best mathematicians survive and live on through their works. Bernhard Riemann was such a mathematician.

In this paper's final analysis, one conclusion and only one remains. Both intuition and rigor are necessary, with them mathematics is very much richer. Individuals usually prefer one over the other, but it is this perfect tension that is the perfect driving force. Forever tied together...

³The author of this paper was so taken back by this part of the lecture, an AHAH moment, the above quote is at best paraphrased, but hopefully the essence is there.

References

- [1] William Cherowitzo. *The Rise of Projective Geometry – Hilbert*. 2008. Lecture, University of Colorado at Denver, Denver, CO, 18 March 2008.
- [2] Keith Devlin. *Sets, Functions, and Logic*. Chapman and Hall/CRC, Boca Raton, London, New York, Washington, D.C., 3rd edition, 2004.
- [3] Harold M. Edwards. *Sets, Functions, and Logic*. Academic Press, Inc., San Diego, CA, 1974.
- [4] Felix Klein. *Development of Mathematics in the Nineteenth Century*. Math Science Press, Brookline, Massachusetts, 1979.
- [5] Detlef Laugwitz. *Bernhard Riemann 1826–1866 Turning Points in the Conception of Mathematics*. Birkhäuser Boston, Boston, Basel, Berlin, 1999. Translated by Abe Shenitzer (from German-language edition).
- [6] Andrew Odlyzko. *The first 100 zeros of the Riemann zeta function, accurate to over 1000 decimal places*. 2008. http://www.dtc.umn.edu/~odlyzko/zeta_tables/: Accessed 31 March 2008.
- [7] Bernhard Riemann. *Collected Papers*. Kendrick Press, Herber City, UT, 2004. Translated from the 1892 edition by Roger Baker, Charles Christenson and Henry Orde.

- [8] Hawking Stephen, editor. *God Created the Integers – The Mathematical Breakthroughs That Changed History*. Running Press, Philadelphia, London, 2nd edition, 2007.
- [9] E. C. Titchmarsh. *The Theory of the Riemann Zeta Function*. University Press, London, 1951.
- [10] (web authors). *Riemann Integral*. 2008.
http://en.wikipedia.org/wiki/Riemann_integral:Accessed
21 March 2008.