Stefan Banach

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1 Introduction

[1] The years just before and after World War I were perhaps the most exciting and influential years in the history of science. In 1901, Henri Lebesgue formulated the theory of measure and in his famous paper *Sur une généralisation de l'intégrale definie*, defining a generalization of the Riemann integral that had important ramifications in modern mathematics. Until that time, mathematical analysis had been limited to continuous functions based on the Riemann integration method. Alfred Whitehead and Bertrand Russell published the Principia Mathematica in an attempt to place all of mathematics on a logical footing, and to attempt to illustrate to some degree how the various branches of mathematics are intertwined.

New developments in physics during this time were beyond notable. Albert Einstein formulated his theory of Special Relativity in 1905, and in 1915 published the theory of General Relativity. Between the relativity theories, Einstein had laid the groundwork for the wave-particle duality of light, for which he was later awarded the Nobel Prize. Quantum mechanics was formulated. Advances in the physical sciences and mathematics seemed to be coming at lightning speed. The whole western world was buzzing with the achievements being made. Along with the rest of Europe, the Polish intellectual community flourished during this time. Prior to World War I, not many Polish scolastics pursued research related careers, focusing instead

[1] Some of this paragraph is paraphrased from Saint Andrews [11].

on education if they went on to the university at all. This was especially true of students in the mathematical sciences. Poland seemed doomed to remain in this state until a plan was enacted by the Polish mathematician Zygmunt Janiszewski, along with several other important people. Janiszewski began

> with the assumption that "Polish mathematicians do not have to be satisfied with the role of followers or customers of foreign mathematical centers," but "can acheive an independent position for Polish mathematics." One of the best ways of achieving this goal, suggested Janiszewski, was for foreign groups of mathematicians to concentrate on the relatively narrow fields in which Polish mathematicians had common interests and – even more importantly – had already made internationally important contributions. These areas included set theory with topology, and the foundations of mathematics (including mathematical logic).[5]

The plan was a radical success, and during the years between World War I and World War II, Poland produced an incredible number of top–ranked mathematicians, more per capita than many other countries. Right around this time, Stefan Banach, perhaps one of the most important and influential mathematicians of the 20th century, arrived on the scene. Largely credited with the first research and publications on functional analysis, Stefan Banach based much of his research on Lebesgue's theory of measure. Banach also made important contributions to many other areas of mathematics. He is considered a national hero in Poland [5]. Here we take a brief look at the life and work of this extraordinary man.

2 The Life and Work of Stefan Banach

Stefan Banach was born on March 30, 1892, at St. Lazarus General Hospital in Cracow [5]. His father's name was Stefan Greczek, his mother is listed on his birth certificate as Katarzyna Banach. Note that the names are different. Stefan Banach was given his father's first name and his mother's last name. They were not married. Of Banach's mother nothing is known, since she gave him up soon after his baptism. Banach tried on several occasions to find out something about her, but his father would never speak of her. He said he had sworn an oath to never reveal the secret. Greczek later married twice and had other sons and daughters. Greczek did not raise his son, leaving him instead in the care of others. It is said that Banach might have been left in the care of his paternal grandmother, but no one knows this with any degree of certainty [5]. Whether or not there is truth in this, it is certain that Banach felt a special closeness with his grandmother, and always kept in close contact with her until her death. It is also certain that Greczek entrusted Banach's care and upbringing to two women, Franciszka Płowa and her daughter Maria. Banach felt affection for these two, regarding them as foster mother and big sister. In the standards of those days he lived in relative comfort, but it seems that he did not have very happy memories of his childhood.

The first person to recognize the fact that Banach had talent was a man named Juliusz Mien, a Frenchman who came to Cracow in 1870 and earned his living as a photographer and translator. He was Maria's guardian, and through her came to know Banach. Mien took Banach under his wing, taught him French, which aided him greatly later in life, and likely encouraged Banach's early inclinations toward mathematics.

It seems that Grezcek never forgot his son and always maintained contact with the Płowas so he could keep an eye on Banach. Grezcek also aided Banach financially whenever possible. Banach never felt any malice toward his father for not raising him, and according to several sources communication bewteen father and son was plentiful, polite and cordial. Although they never really displayed any warmth or affection, Banach's half sister Antonina told Kałuza[5] in an interview that although they did have love for each other, her father and half-brother were never very demonstrative people.

In 1902, at age 10, Banach began attending school at the Henryk Sienkiewicz Gymnasium Number 4 in Cracow. There is not much information regarding Banach's school days, aside from official documents. However, we do know a few things about Banach while he was at the gymnasium. He had two very good friends, Witold Wilkosz, who later became a mathematician, and Marian Albiński. It is through Albiński we have any information at all about Banach during this time. Albiński remembers Banach as a good friend, not without a sense of humor, but generally secretive. Albiński describes both his friends in his memoirs:

> Banach was slim and pale, with blue eyes. He was pleasant in dealings with his colleagues, but outside of mathematics he was not interested in anything. If he spoke at all, he would speak very rapidly, as rapidly as he thought mathematically. He had such an incredible gift for fast thinking and computing that his intelocutors had the impression that he was clairvoyant.
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> Wilkosz was a similar phenomenon. Bewtween the two of them, there was no mathematical problem that they could not speedily tackle. Also, while Banach was faster in mathematical problems, Wilkosz was phenomenally fast in solving problems in physics, which were of no interest to Banach.[5]

There are also some reports that Banach had a large degree of skepticism. It seems he would often ask the school priest, Father Pyłko, questions such as "If the Lord is omnipotent, can he create a rock that he cannot lift himself?"[5]

In 1910, Banach, along with the rest of his class, faced the matura examinations. This might have been a crisis for young Banach. As quoted above, Banach was not interested in anything outside of mathematics. By the time Banach faced the matura, he had eight failing grades [5]. The school board voted to let him graduate, with the deciding vote being that of Father Pyłko, despite Banach's sometimes embarassing questions.

After they gained their matura certificates in 1910, Banach and Wilkosz discussed their plans for the future. Both were mathematically gifted, but they both had the impression that mathematics was so advanced there was not much else to be done, and a research career in mathematics might not be possible nor advisable. They decided to try a different road. In addition to mathematics, Wilkosz also had a talent for languages, so he decided to study Oriental Languages and went on to the Jagiellonian University. However, after two years there he changed his focus to mathematics and gained his Doctor of Philosophy in 1919. Banach decided to study engineering and went to the Lvov Polytechnic in present day Ukraine. Kałuza[5] tells us that Banach laughingly admitted later that he and Wilkosz were somewhat mistaken about the opportunities extant in mathematics.

Banach left Cracow for Lvov without any expections of financial help. Grezcek told him that he had promised to see that Banach got his matura certificate, but from then on he was on his own. He had to work his way through college, so he moved at a somewhat slower pace than most students. In 1914, he passed his half-diploma examination, which means he finished his freshman and sophomore years. Then World War I began, and classes were discontinued for a time. Banach was excused from military service because he was left handed and had weak vision in his left eye. It seems he still wanted to participate though, so he got a job as a foreman on a road crew. Not much else is known about his life during the war. Classes resumed in 1915, and at this point perhaps Banach went back to school. This, however, is uncertain.

In 1916, a chance meeting with a man named Hugo Steinhaus changed the course of Banach's life, both professional and personal. From Kałuza:

> Who was Hugo Dyonizy Steinhaus? He was born in 1887 in the Galician town of Jasło into a family of Jewish intelligenstia (his uncle was a well known politician, member of the Austrian pariament). Although barely five years Banach's senior, his mathematical career was far ahead of the latter's. He obtained formal education in Lvov and then in Göttingen, where in 1911 he received a Ph.D. with David Hilbert for a thesis on trigonometric series. From 1920 to 1941 he was a Professor of Mathematics at Lvov University. Later, following World War II, he moved to the Western Silesian city of Wrocław, where he was a university professor and a member of the Polish Academy of Sciences. During this period he also visited several universities in the United States, inlcuding Notre Dame. Steinhaus, together with Banach, was to be the founder of the Lvov School of Mathematics. His bibliography contains over 170 articles with contributions in Fourier series, orthogonal expansions, linear operators, probability theory, game theory, and applications of mathematics to biology, medicine, electrical engineering, law, and statistics. He was also a well-known popularizer of science.[5]

Steinhaus always called Banach his greatest discovery. In his memoirs Steinhaus tells of the chance meeting with Banach:

> Although Cracow was still formally a fortress, it was already safe to promenade in the evening on Cracow's Planty. During one such walk I overheard the words "Lebesgue measure." I approached the park bench and introduced myself to two young apprentices of mathematics. They told me that they had another companion by the name of Witold Wilkosz, whom they extravagantly praised. The youngsters were Stefan Banach and Otto Nikodym.
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> From then on we would meet on a regular basis, and given the presence in Cracow at that time of Władysław Ślebodziński, Leon Chwistek, Jan Kroć, and Włodzimierz Stozek, we decided to establish a mathematical society.[5]

The Society spoken of here is the Polish Mathematical Society, which will have some importance later on.

In 1920, Banach was married to Łucia Braus, whom he met through Steinhaus. Kałuza tells us that "Banach was very much in love with his wife. She remained his closest and most loyal companion throughout trials and tribulations of the remaining twenty-five years of his life. He called her Lusia."[5]

Steinhaus described a problem to Banach one afternoon that had given him difficulties for quite some time. A few days later Banach had solved it. The problem was concerned with the convergence of partial sums of the Fourier series of an integrable function. Steinhaus was impressed, and after some delay the solution was published in the *Bulletin of the Cracow Academy* in 1918, entitled *Sur la convergance en moyenne de séries de Fourier*. This was Banach's first publication and marked his debut as a mathematician.

From then on, Banach was more or less committed to mathematics. His ideas were original and noteworthy, and he turned out papers at a rather impressive rate. With the publication of his paper on Fourier series, the Polish mathematical community began to take note of this new and unquestionable talent. The paper was written in French, which was the preferred language of the European scientific community. The end of World War I marked the establishment of an independent Polish state, and during this time the Polish mathematical community underwent some rather profound changes in the way it collaborated on research. Banach and Steinhaus, among others, were instrumental in this process. During the years between 1919 and 1929, Banach's research and contributions to this process earned him a reputation as one of Poland's leading mathematicians.

In 1919, Banach turned out his second publication, *Sur la valeur moyenne des fonctions orthogonales*, this time on his own. Banach proves the following theorem (quoted in his own words without proof from [3]):

Soit $\{f_n(t)\}$ une suite compléte de fonctions orthogonales et normées par rapport á l'intervalle $a \leq t \leq b$; autrement dit, une suite dont les termes satisfont á la condition

$$\int_a^b f_i(t) f_k(t) dt = \begin{cases} 1 & \text{pour } i = k \\ 0 & \text{pour } i \neq k \end{cases} \qquad (i, k = 1, 2, \ldots, n, \ldots), \quad (1)$$

sans qu'il soit possible de trouver une fonction $f_0(t)$ telle que la condition (1) soit remplie pour tous les entiers $k \geq 0$. On aura

$$\frac{1}{n} \sum_{k=1}^{n} f_k(t) \to 0 \qquad (\text{pour} \quad n \to \infty) \qquad (2)$$

pour presque tous les t de l'intervalle $\langle a, b \rangle$.

Roughly, this theorem states that the sum of the average of a sequence of orthonormal functions is convergent to zero.

Later that same year, Banach's next paper *Sur l'équation fonctionnelle* $f(x+y) = f(x) + f(y)$ was published. The problem of finding functions satisfying that condition had been the quest of many mathematicians, beginning with Louis Augustin Cauchy. Banach proved a result that was stronger than many that had come before, while at the same time being a simpler and more elegant proof. These two papers alone would have secured Banach's repuation in the realm of functions of real variables, but he didn't stop there. Indeed, these two papers marked the beginning of a long stream of research and publications in that area of mathematics.

The year 1920 was a busy one for Banach. In addition to getting married, Banach was offered a job as an assistant at Lvov Polytechnic. This was his first paying job in academia. Also, in addition to his other writings, namely his fifth paper *Sur les solutions d'une équation fonctionelle de J. Cl. Maxwell* co-authored with Stanisław Ruziewicz, and his sixth, entitled *Sur les fonctions dérivées des fonctions mesurables*, Banach finally wrote his doctoral dissertation. His thesis, entitled *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, deserves attention for two reasons: it earned him his Ph.D. and it contains some of the most important ideas he ever produced. Kałuza writes:

> First of all, it introduces an abstract object that later came to be called a *Banach space*. Banach gave an axiomatic definition of such possible infinite dimensional spaces and introduced the notion of a linear transformation of them. A variety of concrete examples encompassing situations that had previously been considered independently of each other now fell under the unifying umbrella of Banach spaces. The familiar Euclidean, finite-dimensional spaces were obvious special cases, but their structure was relatively simple, and the only linear transformations of them were translations, rotations, and reflections. Much more exciting were the infinite-dimensional Banach spaces of functions important in many areas of classical analysis. Their structure, and the structure of linear transformations (operators) on them, was

> much richer and is not fully understood even today. Although it was Banach's first paper in the area that is now called *functional analysis*, his axiomatization was amazingly complete and, in a sense, final. It should be said that, to some degree, his dissertation brought functional analysis to independent life in a single sweep.[5]

Banach insisted on the completeness of these spaces in his thesis. The reason for this was his assertion that complete spaces are necessary to prove useful theorems. Two such useful and important theorems appear in his thesis. The first states that "the point-wise limit of a sequence of linear and continuous functions is necessarily linear and continuous as well"[5]. The second is known to almost any mathematician. This is the famous *Banach Fixed Point Theorem* for contraction mappings, quoted here from [6]:

> **Banach Contraction Mapping Theorem:** Let $T : X \to X$ be a contraction mapping with contractivity factor α on a complete metric space (X, d) into itself. Then the mapping T possesses exactly one fixed point $u \in X$. Moreover, for any $x \in X$, the sequence $x, T(x), T^2(x), \ldots, T^k(x)$ converges to the point u; that is
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> $$\lim_{k \to \infty} T^k(x) = u.$$

Banach had to overcome some difficulties in order to earn his doctorate. He was primarily a self-taught man, and had never graduated from the Lvov Polytechnic. It took a special legal exemption from the Minister of Education to get Banach admitted to the Masters program, which was a necessity before on could stand for the Ph.D. examinations. Also, in spite of the impressive number of scholarly publications, Banach hated writing. He loved the mental exercise of solving problems, but when it came time to put it all down on paper, he would get bored. His dissertation had been all prepared in his head for quite some time, a fact that his professors were well aware of, but in fact he had to be tricked into getting it written up. Kałuza writes:

> This was recorded by Turowicz:
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> > Professor Ruziewicz instructed one of his assistants to accompany Banach on his frequent visits to the coffee houses, query him on a discrete fashion on his work, and afterwards write down Banach's theorems and proofs. When all of this information was typed out, the notes were presented to Banach, who then edited the text. This is how his Ph.D. dissertation was finally completed.
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> At the age of 28 Banach became a doctor of mathematical sciences. His advisor was Antoni Łomnicki.[5]

The next few years were busy for Banach. In April of 1922, he received his habilitation. In July of that year, he was appointed Professor Extraordinarius at Jan Kazimierz University. Financially, he should have been fairly comfortable, but he took to living somewhat beyond his means, and as a result, accrued some debts. He wrote textbooks during this time to supplement his income. In spite of the drain on his time, he still conducted research and published papers at a rather impressive rate. He published several more papers, including *Sur le problème de la mesure*, wherein two now common notions were first presented; the Banach integral and the Banach generalized limit. These two items are important, but will always play a secondary role to Lebesgue measure, because Lebesgue measure covers a broader range of functions.

In 1924, he published several papers, including the one entitled *Sur la décomposition des ensembles de points en parties respectivement congruentes*, coauthored by Alfred Tarski, that appeared in *Fundamenta Mathematica*. This paper presents what is now known as the Banach–Tarski paradox, which calls into question the axiom of choice. Their statement is basically that given a ball of some size, using the axiom of choice it is possible to decompose the ball into many infinitesimal pieces, and then reassemble two balls exactly the same as the first. The axiom of choice guarantees the existence of the two balls, but provides no details as to their construction. This has some importance because it "was one of the few nontrivial modern mathematical discoveries that could be easily described to lay people"[5]

The years from 1924 to 1929 were happy ones for Banach. He continued to research and publish papers, taught classes, and traveled. He was an honored and well respected man of science. In 1929, together with Steinhaus, Banach founded a new scholarly journal, the *Studia Mathematica*, which would have some importance later on. It is said that Banach's two most important contributions to mathematics during the years between the world wars were the founding of the *Studia Mathematica* and the publication of Banach's most influential work, *Théorie des opérations linéares*, published in 1931.[5]

If Banach was busy in the 1920's, the 1930's must have seemed overwhelming. In addition to his continued publications, in 1932 he began a term as Vice-President of the Polish Mathematical Society. He also continued to conduct classes at Jan Kazimierz University and the Lvov Polytechnic. Banach also published several more papers which "continued to build on the foundations of functional analysis established in the *Theory of Linear Operations* and broadened the trend of applying the discoveries of functional analysis to other areas."[5] Research, teaching, and the accumulation of honors more or less sums up Banach's life during the 1930's.

One thing we have not yet mentioned is Banach research style. He apparently did not care much for working alone in a quiet place. It seems he needed some background noise in order to really concentrate. Mentioned above is a reference to Banach's "frequent visits to the coffee houses"[5]. Banach actually did most of his best work at a place called the Scottish Café, which like him has taken a place in mathematical legend.

Stanisław Ulam, one of Banach's students, explains some of what went on at the Scottish Café in his autobiography *Adventures of a Mathematician*[7]. Ulam describes marble tables that could be written on with a pencil. Banach would write some equation on the table such as $y = f(x)$, and then everyone around the table would sit there drinking coffee and discussing it. We read that some of these sessions lasted literally for days. When Banach worked on a problem, he seems to have been unusually focused, and would think about little else. Indeed, some truly incredible results were accomplished on these marble tables, most of which has been lost. However, some of it was saved in a book Ulam describes[7]. The so-called *Scottish Book* is also mentioned many other places. The website of the Lviv Acturaries (Lviv is a variance of Lvov) tells us of the book:

> In 1935 a large notebook was purchased by Banach and deposited with the head waiter of the "Scottish Cafe". Mathematics questions/problems which after considerable discussions were found suitable were recorded in the "book". Occasional visitors (Henri Leon Lebesgue, John von Neumann, Waclaw Sierpinski) also recorded their problems there. Some of the problems were solved immediately or shortly after they have been posed. Quarter of the problems remain unsolved to this day. When World War II started, Lwow was occupied by the Soviet Union. The last entry into the book was made on May 31, 1941, - less than a month before the war betweeen Germany and Soviet Union began. When the World War came Mazur putted the book in a little box and

buried near the goal post of a certain soccer field in Lviv. The book survived and was found after the war by the son of Banach (who died in 1945) in Lviv. It was given to Steinhaus who in 1956 sent a copy of it to Ulam in US (Los Alamos, New Mexico).

Every problem in the book carries the name of the person who suggested the problem. Frequently a prize is offered for solution of the problem. Prizes range from "two small beers" to "fondue" in Geneva. The book contains about two hundred problems, written mostly in Polish, but also in German, Russian, French and English. The book was translated to English and published in Los Alamos by Ulam in 1957. It came to be known among mathematicians as "The Scottish Book". Later a corrected reprint was made in 1977. In 1981 a version with comments, as well as lectures of "The Scottish Book Conference" was published by Birkhauser publishers (Boston) under the name "The Scottish Book: Mathematics from the Scottish Cafe" edited by R. Daniel Mauldin.[12]

Banach's most important work was in the area of functional analysis. Some of the theorems he proved are regarded as "fundamental" to the study of functional analysis. Some of these are listed in Appendix B, taken from Siddiqi[6].

Just before Word War II began, Banach recieved several honors that confirmed his status in the mathematical community. In April of 1939, he was elected President of the Polish Mathematical Society, "the crowning acheivement of his active involvement in the society he helped to found twenty years earlier."[5] In June of that year, Banach was awarded the Grand Prix by the General Assembly of the Polish Academy of Knowledge. This came with a generous monetary award, equivalent to the average annual stipend of an American University professor. Banach was never able to use that money though, because the war broke out and all bank accounts were frozen. Kałuza writes:

> The offical award ceremony was to take place in October, the beginning of the new academinc year, but the war started on September 1 and no celebration was possible. By October 1 Soviet troops had occupied Lvov.[5]

Banach's vaunted reputation served him well when the Soviets took control of Lvov. They held him in very high regard for his contributions to the mathematical sciences, and he had always maintained good relations with them. It seems that inquiries about him began almost before the dust settled, and he was offered the opportunity to remain in Lvov as Dean of the Physical-Mathematical Faculty and Head of the Department of Mathematical Analysis at Jan Kazimierz University. This was at a time when many other Poles were being deported or generally persecuted. Right around this time, during the first few months of the war, Banach's father and his half-sister Antonina arrived in Lvov trying to escape from the Germans. Apparently, rather than being put out by their arrival, Banach was happy that he could now be of aid to his family and share with them the social status he had earned. He had never been at all bitter or angry at his father, and although he had never been close to Antonina, who was twenty years his junior, during this time he developed the sort of relationship with her that most siblings share. As of this time, Banach's life hadn't really changed much. He continued to teach, research, write, and perform his administrative duties with all the attention he had always given these activities. Unfortunately, this did not last for the duration of the war.

In June of 1941, the German army entered Lvov. The fact that Banach was friendly with the Soviets put him in some danger. The fact that he was a member of the intellectual elite put him in even more danger. Kałuza writes:

> By Himmler's order issued within the so-called *Ausserordentliche Befriedungsaktion* (Extraordinary Pacification Action), the Lvov elite were to be entirely eliminated. In Cracow, the professors had been summarily arrested and deported to concentration camps two years earlier. The liquidaton of the Lvov intelligenstia had been already planned in 1939. Two lists of people to be executed had been prepared. The first contained names of scientists employed by the Polytechnic and the Merchant School; the second, names of university professors. The preparations were

> meticulous, and even the place of executions had been selected in advance. The murderous plans were carried out in secrecy, as opposed to the more public actions in Cracow that caused major reverbations in Western public opinion. The special liquidation commando appeared, entered into action just behind the advancing front-line troops, did his work, and then disappeared. The action appeared to have had no official Wehrmacht sanction.
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> During the night of July 3, 1941, 40 Polish scholars, professors, writers, and other distinguished representatives of the Lvov intelligentsia perished at the hands of the Nazis and the Ukranian nationalists from the S.S. "Nachtigall" battalion. Included in the group were writer and journalist Tadeusz Boy-Żeleński, and Banach's friends and colleagues Włodzimierz Stożek, surgeon Tadeusz Ostrowski, Antoni Łomnicki, and Stanisław Ruziewicz.
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> The executions were only the beginning of a broader action aimed at the destruction of Polish intellectual life in Lvov.[5]

Banach survived the assassinations, and whether or not his name was on the list is uncertain. Yet with his exalted reputation it seems, in the author's opinion, improbable that his name would be absent from this list. Surviving the night of the assassinations of the the Polish intellectual elite was both a blessing and a curse for Banach. Life is almost always better than death, yet the conditions Banach was forced to live under during the Nazi occupation of Lvov were very harsh. The only employment he could secure during the war was as a feeder of lice at the Rudolf Weigl Bacteriological Institute. The job did have a good side. Everyone who worked there received an identification card that allowed them to live in relative security during the Nazi occupation. The identification card allowed them to avoid a great many misunderstandings. At the same time though, it was a miserable way to make a living. Banach's health began to deteriorate. He continued his job at the Institute until the Nazis left Lvov in 1944.

After the Nazis left, the Soviets returned to Lvov. This meant an immediate improvement in the quality of life for Banach and his family. However, the reason for Banach's failing health finally became clear after the Nazis left. He had developed lung cancer, and died before he could retake his place in Lvov society. Kałuza writes:

> Stefan Banach died on August 31, 1945, in Lvov in the apartment at 12 Dwernickiego Street. He was just 53 years old and still bursting with plans for the future. The news about the great success in the U.S. of his former students and collaborators had just begun to filter in from behind the Western front. The newspapers of the period showed the depth of mourning in the Lvov scientific community upon his death. His funeral was attended by hundreds of people. The sidewalks of St. Nicolaus Street, which

he had walked so often in the past, were lined with female students of the former Jan Kazimierz University, each one holding a bunch of flowers. Banach's remains were laid to rest in the family crypt of the Riedls. The tomb is located at the Łyczakowski cemetary, by the no-longer used gate, formerly opening on St. Peter Street. In 1990, his ashes were moved to the Crypt of the Distinguished at the Na Skałce Church in Cracow.

3 Conclusion

Thus ended the life of one of the greatest mathematicians of the 20$^{\text{th}}$ century. The legacy he left us in the form of his work continues to be of tremendous value, and new applications for his theories are being discovered all the time. A search of *Mathematical Reviews* would result in over 11,000 publications with Banach's name in the title, whereas the name David Hilbert yields close to 7,000. Stefan Banach's importance to mathematics cannot be understated. One is curious what else Banach might have accomplished had he lived longer. Some of his theories are not fully understood even today, and research into what he began will likely continue for a long time.

4 A Note on References

The author feels it important to mention that very little material is available in regards to Stefan Banach's life. Of the little that is available, the biography written by Roman Kałuza is the primary source. We have had to determine therefore, that however much reliability this source has, the lack of any other available material has placed a constraint upon us, and the author has had to use this book as his primary source as well. It can be stated with a degree of certainty, however, that nothing more reliable exists.

For the curious, we have compiled a list of Banach's publications in Appendix A. This list does not include the textbooks he wrote, but is limited to his scholarly journals. The collected works of Banach was published in 1979, entitled *Œuvres, avec des commentaires.* All the items on the list in Appendix A can be found there. Appendix B contains what the author feels are some of the more important theorems Banach proved. They are considered fundamental to the study of functional analysis.

5 Appendix A

The Publications of Stefan Banach [3]

1. *Sur la convergence en moyenne de séries de Fourier*, with H. Steinhaus, 1919.
2. *Sur la valeur moyenne es fonctions orthogonales*, 1920.
3. *Sur l'équation fonctionelle* $f(x + y) = f(x) + f(y)$, 1920.
4. *Sur les ensembles de points où la dérivée est infinie*, 1921.
5. *Sur les solutions d'une équation fonctionelle de J. Cl. Maxwell*, with S. Rusiewicz, 1922.
6. *Sur les fonctions dérivées des fonctions mesurables*, 1922.
7. *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales* (Doctoral Thesis), 1922.
8. *An example of an orthogonal development whose sum is everywhere different from the developed function*, 1923.
9. *Sur le problème de la mesure*, 1923.
10. *Sur un théorème de M. Vitali*, 1924.
11. *Sur une classe de fonctions d'ensemble*, 1924.
12. *Un théorème sur les transformations biunivoques*, 1924.
13. *Sur la decómposition des ensembles de points en parties respectivement congruentes*, with A. Tarski, 1924.
14. *Sur les lignes rectifiables et les surfaces dont l'aires est finie*, 1925.
15. *Sur une proriété caractéristique des fonctions orthoganales*, 1925.
16. *Sur le prolongement de certaines fonctionelles*, 1925.
17. *Sur la convergence presque partout de fonctionelles linéaires*, 1926.
18. *Sur une classe de fonctions continues*, 1926.

19. *Sur le principe de la condensation de singularités*, with H. Steihaus, 1927.

20. *Sur certains ensembles de fonctions conduisant aux équations parteilles du second ordre*, 1927.

21. *Sur les fonctions absolument continues des fonctions absolument continues*, 1928.

22. *Sur les fonctionelles linéaires*, 1929.

23. *Sur les fonctionalles linéaires II*, 1929.

24. *Sur une généralisation du problème de la mesure*, with C. Kuratowski, 1929.

25. *Rachunek różniczkowy i całkowy (Calcul différentiel et intégral)* tom I, 1929.

26. *Rachunek różniczkowy i całkowy (Calcul différentiel et intégral)* tom II, 1930.

27. *Sur la convergence forte dans le champ* L^p, with S. Saks, 1930.

28. *Über einige Eigenschaften der lakunären trigonometrischen Reihen*, 1930.

29. *Bemerkung zur Arbeit "Über einige Eigenschaften der lakunären trigonometrischen Reihen"*, 1930.

30. *Über additive Ma* β *funktionen in abstrakten Mengen*, 1930.

31. *Théorème sur les ensembles de première catégorie*, 1930.

32. *Über analytisch darstellbare Operationen in abstrakten Räuman*, 1931.

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6 Appendix B

Important Theorems of Stefan Banach [6]

Stated without proof.

Banach Contraction Mapping Theorem: Let $T : X \to X$ be a contraction mapping with contractivity factor α on a complete metric space (X, d) into itself. Then the mapping T possesses exactly one fixed point $u \in X$. Moreover, for any $x \in X$, the sequence $x, T(x), T^2(x), \ldots, T^k(x)$ converges to the point u; that is

$$\lim_{k \to \infty} T^k(x) = u.$$

The Hahn-Banach Theorem: Let X be a real vector space, M a subspace of X, and p a real function defined on X satisfying the following conditions:

1. $p(x + y) \leq p(x) + p(y)$,

2. $p(\alpha x) = \alpha p(x)$,

$\forall x, y \in X$ and positive real α. Further, suppose that f is a linear functional on M such that $f(x) \leq p(x) \quad \forall x \in M$. Then there exists a linear functional F defined on X for which $F(x) = f(x) \quad \forall x \in M$ and $F(x) \leq p(x) \quad \forall x \in X$. In other words, there exists and extension F of f having the property of f.

Topological Hahn-Banach Theorem: Let X be a normed space, M a subspace of X, and f a bounded linear functional on M. Then there exists a bounded linear functional F on X such that

1. $F(x) = f(x) \quad \forall x \in M$,

2. $\|F\| = \|f\|$.

In other words, there exists an extension F of f which is also bounded linear and preserves the norm.

The Banach-Alaoglu Theorem: Suppose X is a normed vector space and X^* is its dual. Then the closed unit sphere $S_1^* = \{f \in X^*/\|f\| \leq 1\}$ is compact with respect to the weak* topology.

The Banach-Steinhaus Theorem: Let X be a Banach space, Y a normed space and $\{T_i\}$ a sequence of bounded linear operators over X into Y such that $\{T_i(x)\}$ is a bounded subset of Y for all $x \in X$. Then $\{\|T_i\|\}$ is a bounded subset of real numbers, i.e., $\{T_i\}$ is a bounded sequence in the normed space $\mathcal{B}[X, Y]$.

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