Geometric Transformations

Definitions

Def: *f* is a ***mapping*** (function) of a set A into a set B if for every element *a* of A there exists a unique element *b* of B that is paired with *a*; this pairing is denoted by $f(a) = b$. The set A is called the ***domain*** of *f*, and the set B is called the ***codomain*** of *f*. We use the notation $f\colon A \to B$ to denote a mapping.

Def: Given a mapping $f\colon A \to B$, the ***range*** of *f* is

$$\{b \mid b = f(a),\ a \in A\}.$$

The range is a subset of the codomain B.

Def: If b is an element of the range of *f*, and a an element of the domain of *f* for which f(a) = b, then b is called the ***image*** of a under *f*.

Definitions

Def: A mapping *f*: A $\rightarrow$ B is ***onto*** B if the range of *f* = B, the codomain of *f*. (surjection)

Def: A mapping *f*: A $\rightarrow$ B is a ***one-to-one*** mapping if each element of the range of f is the image of exactly one element of A. That is, if f(a) = f(b) then a = b. (injection)

Def: A mapping *f*: A $\rightarrow$ B is a ***transformation*** if it is onto and one-to-one. (bijection)

Product of transformations (composition of functions), identity, inverses.

Examples

f: $\mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2x + 3$.

f is a mapping from the domain of real numbers to the codomain of real numbers.

f is a one-to-one mapping since
if $f(a) = f(b)$ then $2a+3 = 2b+3 \Rightarrow a = b$.

f is an onto mapping.
Let r be any element of the codomain, we must find an a in the domain so that $f(a) = r$. But this just means, solve for a: $2a+3 = r$, i.e., $a = \frac{1}{2}(r-3)$. Since r is a real number, a is a real number. So any element of the codomain is in the range and f is onto.

f is a transformation.

Examples

f: $\mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^2$.

f has domain and codomain $\mathbb{R}$.

f is **not** one-to-one.
If $2(a^2) = 2(b^2)$, then $a^2 = b^2$, but this does not mean that a = b, since we could have a = -b as well. Specifically f(2) = f(-2) but 2 and -2 are not equal.

f is **not** onto.
Consider -1 which is in the codomain. There is no real number a with $2a^2 = -1$, so -1 is not in the range of f.

Geometric Transformations

By identifying the real numbers with points on the line (the real number line), the previous two examples can be thought of as mappings from the points on the line to other points on the same line. The first example was a transformation, but the second one was not.

Similarly, by identifying points in the plane with coordinate pairs, we can define transformations from the points in the plane to itself. These examples give such:

$$f(x,y) = (3x + 2, 4y - 5), \quad \text{or}$$

$$f(x,y) = (2x - y + 3, x + 4y - 2)$$

Geometric Transformations

Geometric transformations serve several useful roles in the study of geometries.

1. Groups of transformations can be used to classify geometries.
2. Properties of configurations that remain unchanged after applying a transformation (***invariants*** of the transformation) are generally the significant properties that one wants to study.
3. Transformations provide the formal groundwork underlying Euclid's loose concept of superimposition.
4. Transformations allow motion to enter into the discussion of an otherwise static subject.
5. Transformations provide the link between Geometry and Abstract Algebra.

Composition of Transformations

Def: Given a transformation f: A $\rightarrow$ B and a transformation g: B $\rightarrow$ C, we can define a transformation h: A $\rightarrow$ C, called the ***composition*** of f and g, by $h(a) = g(f(a))$, often written as $h = gf$.

Def: The ***identity*** transformation I: A $\rightarrow$ A on any set A is defined by $I(a) = a$ for all a in A.

Def: Given a transformation f: A $\rightarrow$ B, the ***inverse*** transformation of f, denoted by f^{-1} is the transformation f^{-1}: B $\rightarrow$ A which has the property that $ff^{-1} = f^{-1}f = I$.

Example

Consider the transformations of the plane given by
$f(x,y) = (x-1, y+2)$ and $g(x,y) = (2y, -x)$.

The composition $gf(x,y) = g(x-1,y+2) = (2y+4, -x+1)$
and the composition $fg(x,y) = f(2y, -x) = (2y-1, -x+2)$.

The identity transformation in this case is $I(x,y) = (x,y)$.

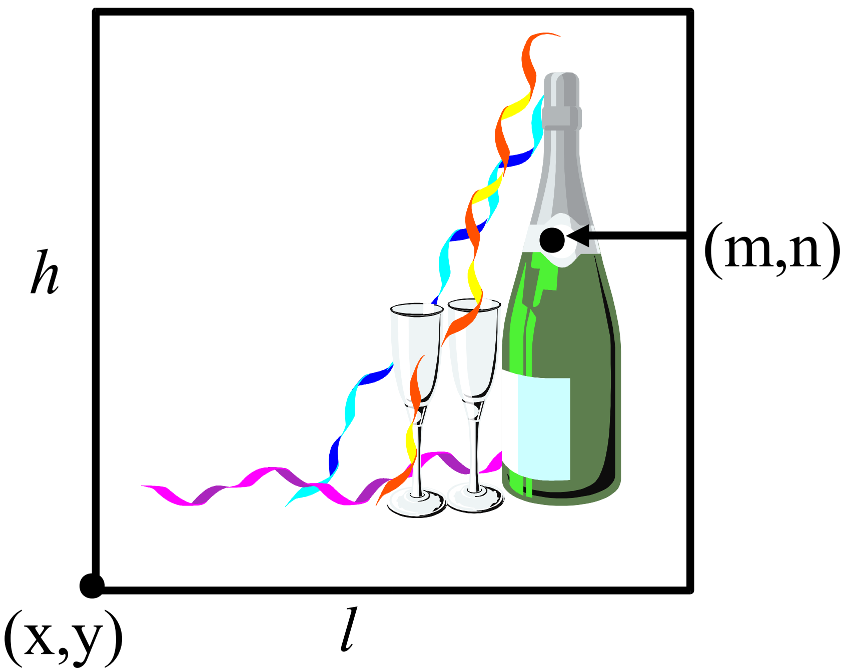
The inverse of f is given by $f^{-1}(x,y) = (x+1, y-2)$ since
$ff^{-1}(x,y) = f(x+1, y-2) = (x, y)$ and $f^{-1}f(x,y) = f^{-1}(x-1, y+2) = (x,y)$.

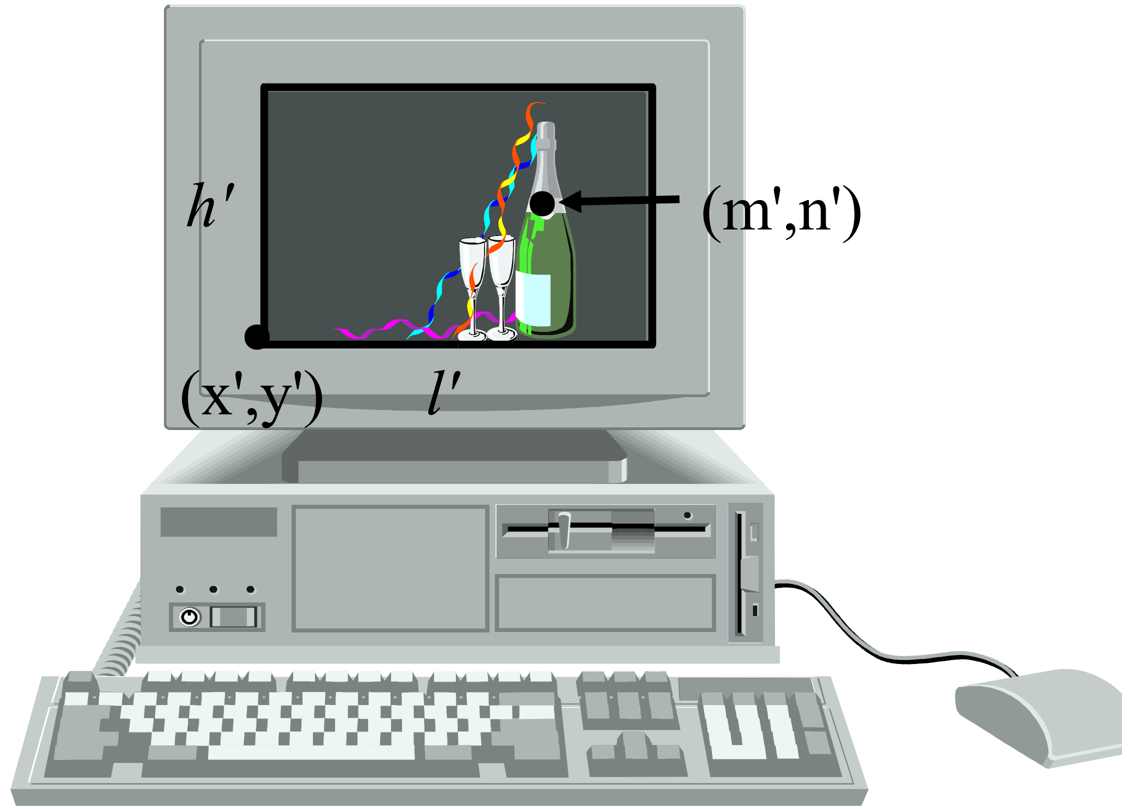
What is the inverse of g?

$g^{-1}(x, y) = (-y, \frac{1}{2}x)$

An Application: Computer Graphics

World Coordinates to Screen Coordinates





$$m' = \left(\frac{l'}{l}\right)(m - x) + x'$$

$$n' = \left(\frac{h'}{h}\right)(n - y) + y'$$

Groups

Def: A ***group*** is a set G together with a binary operation $\otimes$ (a binary operation is a mapping from $G \times G \to G$) such that

1. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all a, b, c in G.
2. There is a unique element I in G so that
   $a \otimes I = I \otimes a = a$ for all a in G. I is called the identity.
3. For each a in G, there exists a unique a^{-1} in G so that
   $a \otimes a^{-1} = a^{-1} \otimes a = I$. a^{-1} is called the inverse of a.

Def: A ***group of transformations*** is a non-empty set of transformations of a set which form a group under the binary operation of composition.

Examples

Some common examples of groups:

1. The real numbers under addition.
2. The rational numbers under addition.
3. The integers under addition.
4. The real numbers except for 0, under multiplication.
5. The rational numbers except for 0, under multiplication.
6. 2x2 matrices over the reals under matrix addition.
7. 2x2 non-singular matrices over the reals under matrix multiplication.

Some non-groups:

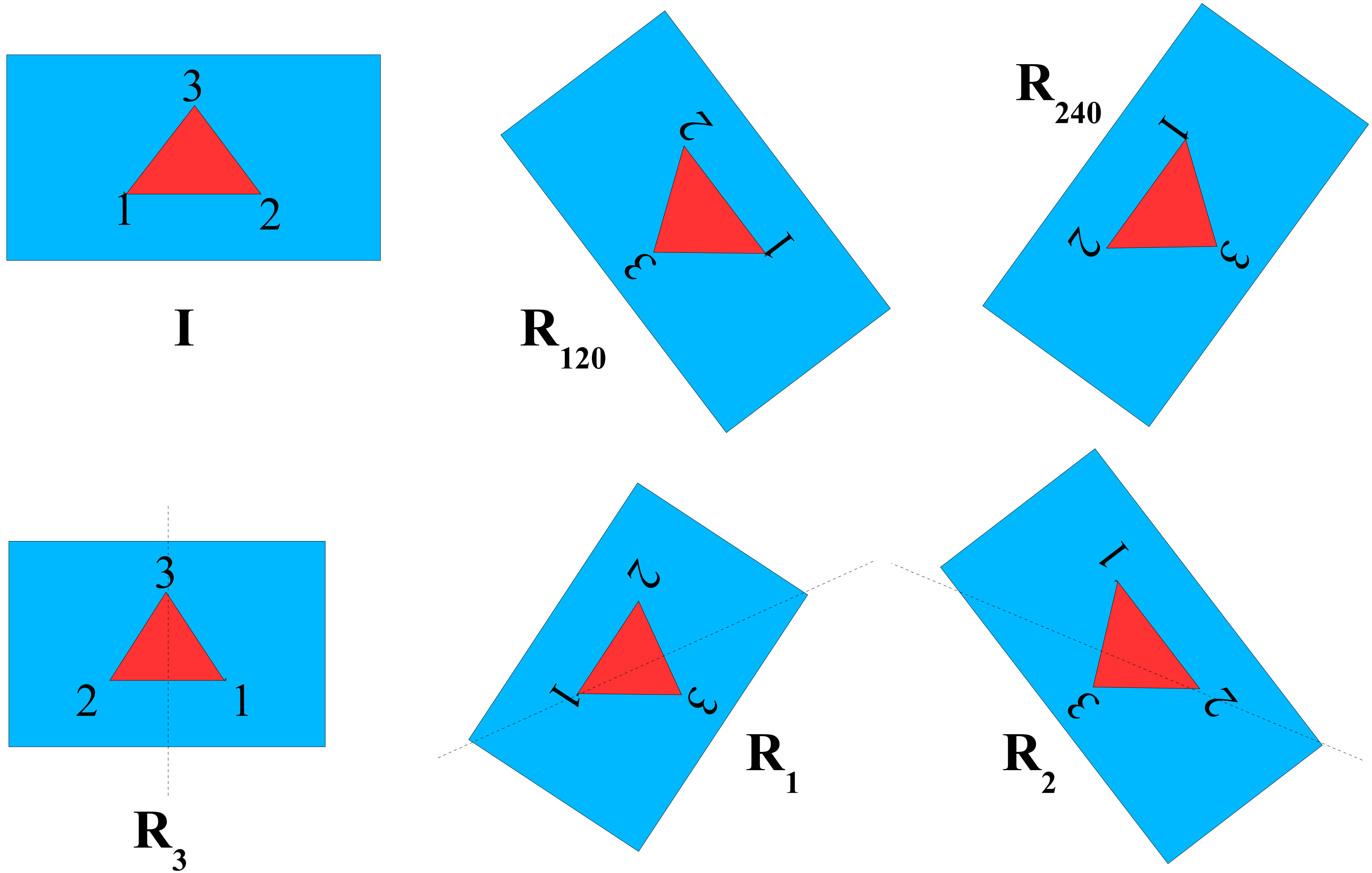
1. The reals under multiplication.
2. The natural numbers under addition.

Symmetries of an Equilateral Triangle

It is easier to represent these transformations using permutation notation.

Two line form	One line form	Cyclic form
$\boldsymbol{I} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$(1\ \ 2\ \ 3\ )$	$(1)(2)(3)$
$\boldsymbol{R_1} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$(1\ \ 3\ \ 2\ )$	$(1)(2\ 3) = (2\ 3)$
$\boldsymbol{R_2} := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$(3\ \ 2\ \ 1\ )$	$(1\ 3)(2) = (1\ 3)$
$\boldsymbol{R_3} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$(2\ \ 1\ \ 3\ )$	$(1\ 2)(3) = (1\ 2)$
$\boldsymbol{R_{120}} := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$(3\ \ 1\ \ 2\ )$	$(1\ 3\ 2)$
$\boldsymbol{R_{240}} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$(2\ \ 3\ \ 1\ )$	$(1\ 2\ 3)$

Symmetries of an Equilateral Triangle



Symmetries of Triangle form a Group

The symmetries of an equilateral triangle form a group under composition. To see this we write out the "multiplication table".

	I	R_1	R_2	R_3	R_{120}	R_{240}
I	I	R_1	R_2*	R_3	R_{120}	R_{240}
R_1	R_1	I	R_{120}	R_{240}	R_2	R_3
R_2	R_2	R_{240}	I	R_{120}	R_3	R_1
R_3	R_3	R_{120}	R_{240}	I	R_1	R_2
R_{120}	R_{120}	R_3	R_1	R_2	R_{240}	I
R_{240}	R_{240}	R_2	R_3	R_1	I	R_{120}

* Note typo in text

sym

sub

Table

Symmetries of Triangle form a Group

To fill out the entries in this table we perform the following calculations:

$$I := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$R_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$R_2 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$R_3 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$R_{120} := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$R_{240} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

To get R_1R_2 (entry in row R_2 column R_1):

For each element do R_2 first and then R_1 to the result -

$R_2(1) = 3, \; R_1(3) = 2$ so $R_1R_2(1) = 2$

$R_2(2) = 2, \; R_1(2) = 3$ so $R_1R_2(2) = 3$

$R_2(3) = 1, \; R_1(1) = 1$ so $R_1R_2(3) = 1$

so $\mathbf{R_1R_2 = R_{240}}$

In cyclic notation $(2\ 3)(1\ 3) = (1\ 2\ 3)$

Table

Symmetries of Triangle form a Group

To check that this is a group we observe the following:

1) Composition is a binary operation on this set – there are no empty cells in the table.

2) Composition is associative – true in general for composition of functions (mappings).

3) I is the identity element – check first row and column.

4) Every element has a unique inverse – there is exactly one I in each row and column of the table. In particular

$I^{-1} = I$, $R_1^{-1} = R_1$, $R_2^{-1} = R_2$, $R_3^{-1} = R_3$, $R_{120}^{-1} = R_{240}$ and $R_{240}^{-1} = R_{120}$.

Subgroups

A ***subgroup*** of a group is a subset of the elements of the group which form a group themselves with the same identity element and the same binary operation.

In our example, the elements I, R_{120} and R_{240} form a subgroup.

Another subgroup is given by {I, R_1} for instance.

This can be seen from the multiplication table.

The full group and the subset consisting of just the identity are always subgroups, but they are known as ***improper*** subgroups. The remaining ones are called ***proper***.

Symmetries of Triangle form a Group

Subgroup {I, R_{120}, R_{240}} indicated in green. {I, R_1} in red.

	I	R_1	R_2	R_3	R_{120}	R_{240}
I	I	R_1	R_2	R_3	R_{120}	R_{240}
R_1	R_1	I	R_{120}	R_{240}	R_2	R_3
R_2	R_2	R_{240}	I	R_{120}	R_3	R_1
R_3	R_3	R_{120}	R_{240}	I	R_1	R_2
R_{120}	R_{120}	R_3	R_1	R_2	R_{240}	I
R_{240}	R_{240}	R_2	R_3	R_1	I	R_{120}

Dihedral Groups

The subgroup $\{I, R_{120}, R_{240}\}$ is called the ***rotational*** subgroup. All of the other regular polygons have symmetry groups which contain a large (relative to the sizes of the other proper subgroups) rotational subgroup.

A ***Dihedral group*** is the symmetry group of a regular polygon.

We denote the symmetry group of a regular polygon with n sides by $\mathbf{D_{2n}}$ (some authors use D_n). Thus, the group we have just studied is D_6.

Isometry

In some geometries the concept of distance is used. In these geometries we may define a special type of transformation.

An ***isometry*** is a transformation which preserves distance.

Euclidean geometry is a geometry with distance. In the Euclidean plane the distance between two points (x_1,y_1) and (x_2,y_2) is given by:

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

A ***motion*** is an isometry from a set onto itself. We are interested in the motions of the Euclidean plane.

Translations

A ***translation*** $\tau_{a,b}$ is the transformation $\tau_{a,b}(x,y) = (x+a, y+b)$, for any pair of real numbers a and b.

A translation is an isometry.

Let (x_1,y_1) and (x_2,y_2) be two points at distance d. Then $\tau_{a,b}(x_1,y_1) = (x_1+a, y_1+b)$ and $\tau_{a,b}(x_2,y_2) = (x_2+a, y_2+b)$ so, the distance between the image points is:

$$d' = \sqrt{(x_1+a-(x_2+a))^2+(y_1+b-(y_2+b))^2}$$
$$= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = d\,.$$

Translations

Translations are easily visualized using vectors.
Using vector addition we have:

$$\tau_{a,b}(x,y) = (x+a, y+b) = (x,y) + (a,b).$$

Thus, the translation $\tau_{a,b}$ can be thought of as adding the vector (a,b) to every point of the plane.

Several properties of translations are now obvious:

1. A translation maps a line segment to a parallel line segment.
2. The vector determined by a point and its image under a translation is always the same (same length and direction).
3. The inverse of a translation is a translation. $\tau_{a,b}^{-1} = \tau_{-a,-b}$.
4. The composition of two translations is another translation.

$$\tau_{a,b}\,\tau_{c,d} = \tau_{a+c,\, b+d}$$

Translations

Definition: A group G in which ab = ba for all a,b in G is called an ***abelian*** (or ***commutative***) group.

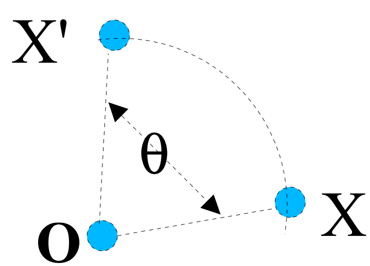
Most of the examples of groups involving numbers were abelian groups. Transformation groups are in general non-abelian groups.

The set of translations of the Euclidean plane form an abelian group of transformations with identity $\tau_{0,0}$. Naturally, this is called the ***translation group*** of the plane.

Rotations

A second basic type of motion of the plane is called a ***rotation***.

A rotation has a fixed center O (a point which doesn't move) and all other points are rotated by an angle θ about O.
(By convention, positive angles correspond to counterclockwise rotations).

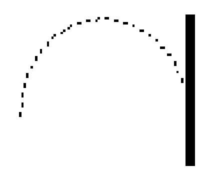


We will show that rotations are isometries when we introduce the analytic form of a rotation.

Rotations

Some properties of rotations:

1. Under a rotation a segment is usually not parallel to its image – but there is an exception.



2. The inverse of a rotation is a rotation with the same center and angle equal in size but opposite in sense.

3. The composition of two rotations with the same center is another rotation with that center.

4. The set of rotations with the same center form an abelian group.

Translations and Rotations

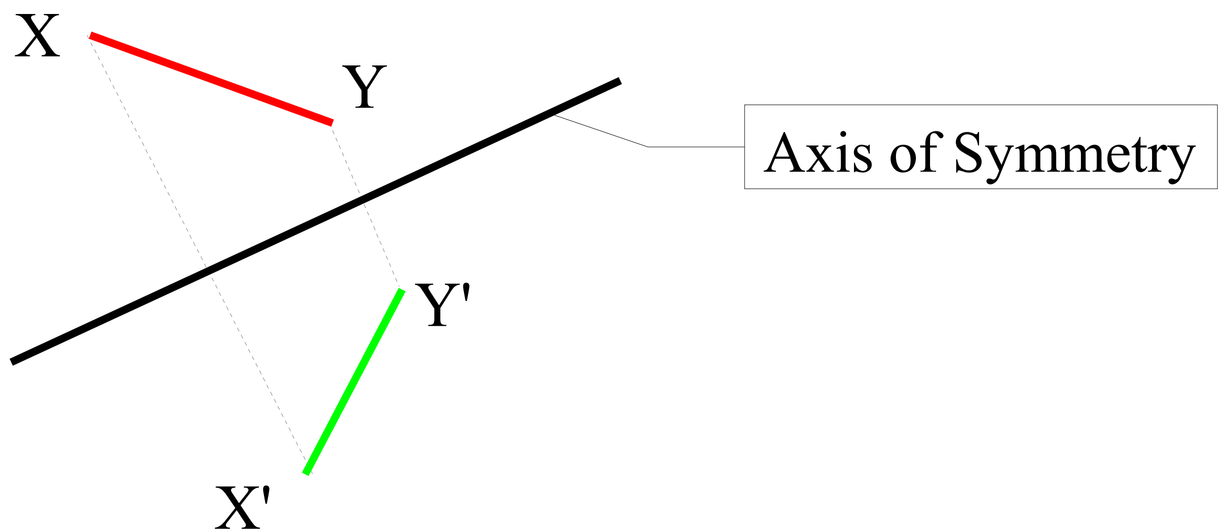
The set of all translations and rotations is known as the set of ***rigid motions*** or displacements of the plane.

The rigid motions form the ***group of rigid motions*** of the plane.

Reflections

A third basic type of motion is the ***reflection***.

A reflection in the plane is defined with respect to a line called the ***axis of symmetry***. Points on the axis of symmetry are fixed. Points off the axis are mapped to points so that the axis of symmetry is the perpendicular bisector of the line segment joining a point and its image. (Think of mirror images with the axis as the mirror).



Reflections

Properties of reflections:

1. A line segment is usually not parallel to its reflected image – but it may be.

2. The inverse of a reflection is the same reflection.

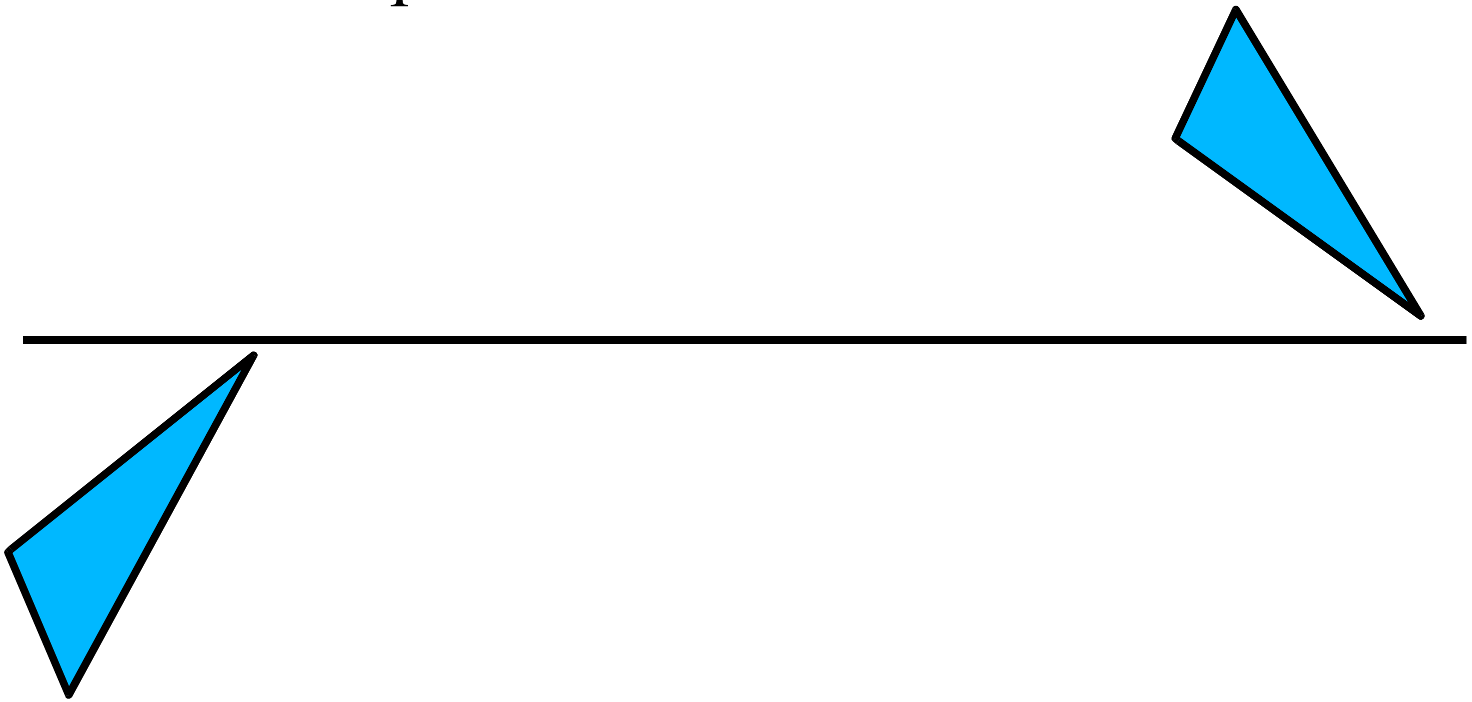
3. The composition of two reflections with the same axis is not a reflection (it is the identity mapping, which is not a reflection).

4. It is generally not possible to "slide" a figure to coincide with its reflected image.

Glide Reflections

In order to form a group of transformations which includes the rigid motions and the reflections, we need to introduce another type of motion, the ***glide reflection***.

A ***glide reflection*** is the composition of a reflection with a translation parallel to the axis of the reflection.



No Quiz Wednesday

Instead of a quiz next Wednesday I would like you to hand in the following homework problems:

pg. 20	# 21
pg. 24	# 9
pg. 30	# 18
pg. 33	# 1

Equations for Rigid Motions

Translations:

We have already described the translations $\tau_{a,b}(x,y) = (x',y')$ where

$$x' = x + a$$
$$y' = y + b.$$

Example: The image of the line $3x + 4y - 5 = 0$ under the translation $\tau_{2,-1}$ is,

$$0 = 3(x' - 2) + 4(y' + 1) - 5 = 3x' - 6 + 4y' + 4 - 5 = 3x' + 4y' - 7.$$

Example: In general, the image of line $cx + dy + e = 0$ under the translation $\tau_{a,b}$ is,

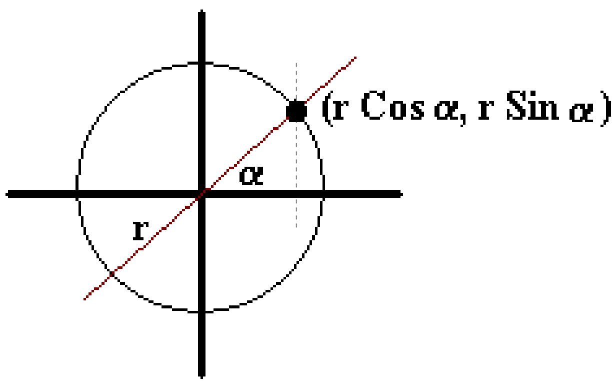
$$0 = c(x'-a) + d(y'-b) + e = cx' - ac + dy' - bd + e$$
$$= cx' + dy' + e - ac - bd.$$

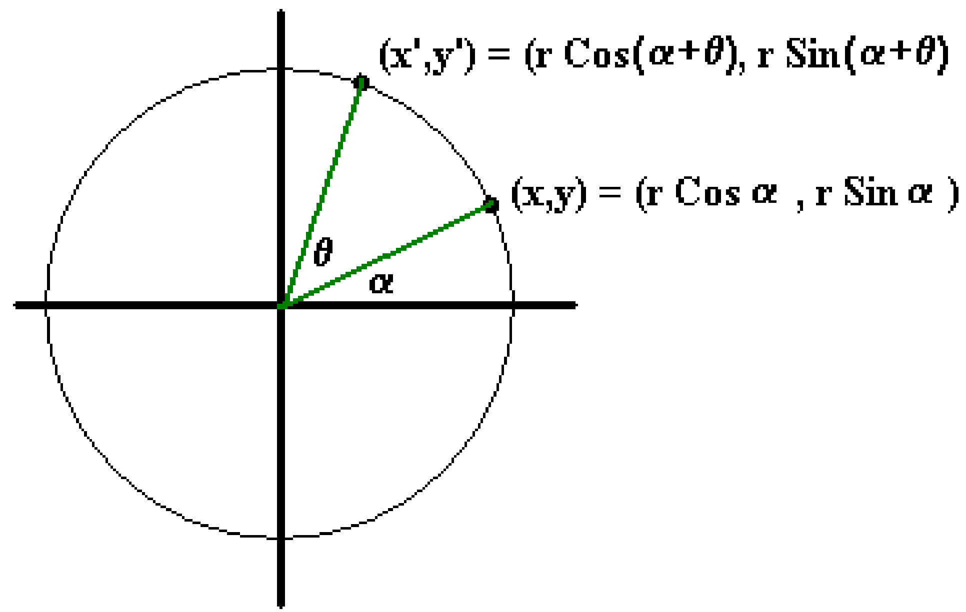
Equations for Rigid Motions

Rotations:

We start with a special case where the center of the rotation is the origin.

Recall,





$$x' = r\,\mathrm{Cos}(\alpha+\theta) = r\,\mathrm{Cos}\,\alpha\,\mathrm{Cos}\,\theta - r\,\mathrm{Sin}\,\alpha\,\mathrm{Sin}\,\theta = x\,\mathrm{Cos}\,\theta - y\,\mathrm{Sin}\,\theta$$

$$y' = r\,\mathrm{Sin}(\alpha+\theta) = r\,\mathrm{Cos}\,\alpha\,\mathrm{Sin}\,\theta + r\,\mathrm{Sin}\,\alpha\,\mathrm{Cos}\,\theta = x\,\mathrm{Sin}\,\theta + y\,\mathrm{Cos}\,\theta$$

Equations for Rigid Motions

Rotations: To get the general form, rotation about an arbitrary point (h,k), we first translate the center of rotation to the origin, do the rotation there, and then translate back to the original setting.

First, use the translation $\tau_{-h,-k}(x,y) = (x'', y'')$, then use the formula for rotation about the origin to get the image (x''', y''') where

$$x''' = x'' \cos\theta - y'' \sin\theta = (x-h) \cos\theta - (y-k) \sin\theta \quad \text{and}$$
$$y''' = x'' \sin\theta + y'' \cos\theta = (x-h) \sin\theta + (y-k) \cos\theta .$$

Finally, translate back with the translation $\tau_{h,k}(x''',y''') = (x',y')$ to get

$$x' = (x-h) \cos\theta - (y-k) \sin\theta + h$$
$$y' = (x-h) \sin\theta + (y-k) \cos\theta + k.$$

Rotations are Isometries

Having the rotation formula we can now prove that a rotation is an isometry.

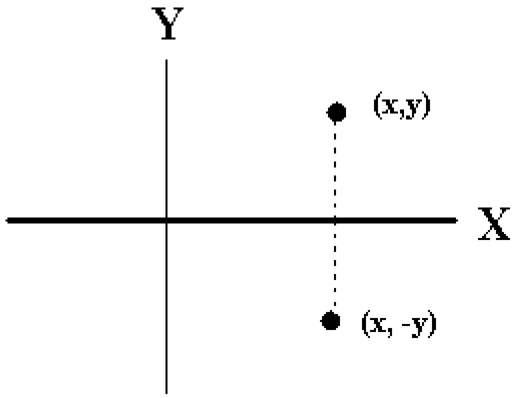
Let (x_1,y_1) and (x_2,y_2) be two arbitrary points at distance d. We apply the rotation with center (h,k) and angle θ to each and calculate the distance d' between the image points.

$$(x_1-h)\cos(\theta)-(y_1-k)\sin(\theta)+h-((x_2-h)\cos(\theta)-(y_2-k)\sin(\theta)+h)$$
$$=(x_1-x_2)\cos(\theta)-(y_1-y_2)\sin(\theta)$$
$$(x_1-h)\sin(\theta)+(y_1-k)\sin(\theta)+k-((x_2-h)\sin(\theta)+(y_2-k)\sin(\theta)+k)$$
$$=(x_1-x_2)\sin(\theta)+(y_1-y_2)\cos(\theta)$$
$$d'=\sqrt{((x_1-x_2)\cos(\theta)-(y_1-y_2)\sin(\theta))^2+((x_1-x_2)\sin(\theta)+(y_1-y_2)\cos(\theta))^2}$$
$$((x_1-x_2)\cos(\theta)-(y_1-y_2)\sin(\theta))^2$$
$$=(x_1-x_2)^2\cos^2(\theta)-2(x_1-x_2)(y_1-y_2)\sin(\theta)\cos(\theta)+(y_1-y_2)^2\sin^2(\theta)$$
$$((x_1-x_2)\sin(\theta)+(y_1-y_2)\cos(\theta))^2$$
$$=(x_1-x_2)^2\sin^2(\theta)+2(x_1-x_2)(y_1-y_2)\sin(\theta)\cos(\theta)+(y_1-y_2)^2\cos^2(\theta)$$
$$d'=\sqrt{(x_1-x_2)^2(\cos^2(\theta)+\sin^2(\theta))+(y_1-y_2)^2(\cos^2(\theta)+\sin^2(\theta))}$$
$$d'=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=d$$

Equations for Reflections

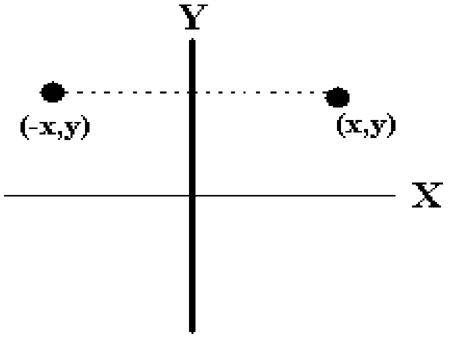
We obtain the formulas for reflections about a line in stages.

Special case: Reflection about the x-axis



$$x' = x$$
$$y' = -y$$

Special case: Reflection about the y-axis

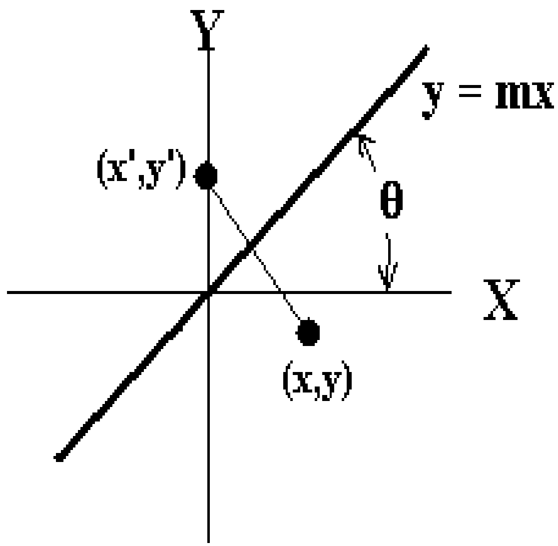


$$x' = -x$$
$$y' = y$$

Special case: Reflection about the line y = mx

To obtain this formula, we rotate with center (0,0) so that the line y = mx is rotated to the x-axis, reflect and then rotate back.

Equations for Reflections



With $m = \tan\theta$, rotate by angle $-\theta$. The point (x,y) becomes (x'', y'') where

$$x'' = x \text{ Cos } (-\theta) - y \text{ Sin } (-\theta)$$
$$y'' = x \text{ Sin } (-\theta) + y \text{ Cos } (-\theta)$$

Reflect thru the x-axis to get (x''',y''') where

$$x''' = x \text{ Cos } (-\theta) - y \text{ Sin } (-\theta)$$
$$y''' = -x \text{ Sin } (-\theta) - y \text{ Cos } (-\theta)$$

Finally, rotate by angle θ to return to the original position and get

$$x' = (x \text{ Cos } (-\theta) - y \text{ Sin } (-\theta)) \text{ Cos } (\theta) - (-x \text{ Sin } (-\theta) - y \text{ Cos } (-\theta)) \text{ Sin } (\theta)$$
$$y' = (x \text{ Cos } (-\theta) - y \text{ Sin } (-\theta)) \text{ Sin } (\theta) + (-x \text{ Sin } (-\theta) - y \text{ Cos } (-\theta)) \text{ Cos } (\theta)$$

Equations for Reflections

We simplify this result, recalling the trig identities:

$$\text{Cos}(-\theta) = \text{Cos}(\theta) \qquad \text{Sin}(-\theta) = -\text{Sin}(\theta)$$

$$\text{Cos}(2\theta) = \text{Cos}^2(\theta) - \text{Sin}^2(\theta) \qquad \text{Sin}(2\theta) = 2\text{Sin}(\theta)\text{Cos}(\theta)$$

$$\begin{aligned} x' &= (x\,\text{Cos}(-\theta) - y\,\text{Sin}(-\theta))\,\text{Cos}(\theta) - (-x\,\text{Sin}(-\theta) - y\,\text{Cos}(-\theta))\,\text{Sin}(\theta) \\ &= x\,[\text{Cos}^2(\theta) - \text{Sin}^2(\theta)] + y[2\text{Sin}(\theta)\text{Cos}(\theta)] \\ &= x\,\text{Cos}\,2\theta + y\,\text{Sin}\,2\theta \end{aligned}$$

$$\begin{aligned} y' &= (x\,\text{Cos}(-\theta) - y\,\text{Sin}(-\theta))\,\text{Sin}(\theta) + (-x\,\text{Sin}(-\theta) - y\,\text{Cos}(-\theta))\,\text{Cos}(\theta) \\ &= x[2\text{Sin}(\theta)\text{Cos}(\theta)] + y[\text{Sin}^2(\theta) - \text{Cos}^2(\theta)] \\ &= x\,\text{Sin}\,2\theta - y\,\text{Cos}\,2\theta. \end{aligned}$$

Note: If $\theta = 90^{\circ}$, $x' = x(-1) + y(0) = -x$ and $y' = x(0) - y(-1) = y$ in agreement with the formula for reflection about the y-axis.

Equations for Reflections

General Case: y = mx + b

If $m \neq 0$ we may proceed as follows: The point where the line y = mx + b intersects the x-axis is (-b/m, 0). If we use the translation $\tau_{b/m,0}$, this line will be translated to y'' = mx''. We then apply the reflection formula, and then translate back to the original position.

This gives the formula,

$$x' = (x + b/m) \text{ Cos } 2\theta + y \text{ Sin } 2\theta - b/m$$

$$y' = (x + b/m) \text{ Sin } 2\theta - y \text{ Cos } 2\theta$$

or

$$x' = x \text{ Cos } 2\theta + y \text{ Sin } 2\theta + b/m \text{ Cos } 2\theta - b/m$$

$$y' = x \text{ Sin } 2\theta - y \text{ Cos } 2\theta + b/m \text{ Sin } 2\theta$$

If m = 0, then we can use the translation $\tau_{0,-b}$ to move the line to the x-axis, apply the reflection formula for the x-axis, and then use the translation $\tau_{0,b}$ to move back to the original position.

Equations for Reflections

This would give the formula:

$$x'' = x,\ y'' = y - b \quad \rightarrow \quad x''' = x,\ y''' = b - y \quad \rightarrow$$

$$x' = x$$
$$y' = 2b - y$$

These two formulas may be combined into a single formula (as is done in the text) by using the parameter d = distance between the lines y = mx and y = mx + b. But we won't derive that version.

Theorem 2.2

If a transformation is a plane motion, then it has equations of the form:

$$\begin{aligned} x' &= ax + by + c \\ y' &= (-bx + ay) + d \end{aligned} \qquad \text{(I)}$$

or

$$\begin{aligned} x' &= ax + by + c \\ y' &= -(-bx + ay) + d \end{aligned} \qquad \text{(II)}$$

for a, b, c and d real numbers satisfying $a^2 + b^2 = 1$.

Pf: From the formulas we have derived we see that translations are of type (I) with $a = 1$, $b = 0$; rotations are of type (I) with $a = \text{Cos}\ \theta$ and $b = -\text{Sin}\ \theta$; reflections are of type (II) with $a = \text{Cos}\ 2\theta$ and $b = \text{Sin}\ 2\theta$; and glide reflections are of type (II) with $a = \text{Cos}\ 2\theta$ and $b = \text{Sin}\ 2\theta$.

Theorem 2.3

The converse of Theorem 2.2 is also true, but we will not provide the proof.

If a transformation has equations of the form:

$$x' = ax + by + c$$
$$y' = (-bx + ay) + d$$

or

$$x' = ax + by + c$$
$$y' = -(-bx + ay) + d$$

for a, b, c and d real numbers satisfying $a^2 + b^2 = 1$, then it is a plane motion.

Matrix Form of Equations

Recall from Linear Algebra:

$$(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy).$$

As the images of many of the motions have this form, we can represent them in matrix form. For example:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	**Identity**	$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$	**General rotation about the origin**
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	**Reflection about the y-axis**	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	**Rotation of 90^{o} about origin**
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	**Reflection about the x-axis**	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	**Rotation of 180^{o} about origin**
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	**Reflection about the line y = x**	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	**Rotation of 270^{o} about origin.**

Matrix Form of Equations

Using matrix multiplication, several points can be dealt with at one time when a matrix can be used to represent the transformation.

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \\ ax_3 + cy_3 & bx_3 + dy_3 \end{pmatrix}$$

Matrix multiplication can also be used to determine the composition of two transformations which are given by matrices.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Rotate by 90° followed by rotate by 180° gives rotate by 270°

More Transformations

Other transformations (which are not isometries) can also be represented by 2×2 matrices (which are non-singular, i.e. det ≠ 0).

Some common ones are:

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	**Scale change in x direction**
$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	**Shear in y direction**
$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	**Unequal scaling change**

Translations

Unfortunately translations can not be done with 2×2 matrices.

However, a slight modification of the way we write point coordinates permits us to write translations (and all the other transformations we've seen) using 3×3 matrices.

$$(x, y, 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} = (x+a, y+b, 1).$$

Direct and Opposite Motions

In the text you will find:

Definition: A ***direct*** motion is a plane motion which is the product of an even number of reflections.

An ***opposite*** motion is a plane motion which is the product of an odd number of reflections.

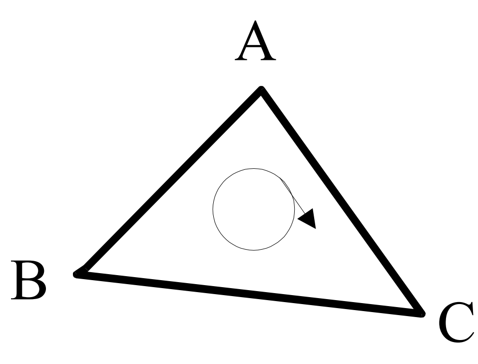
... it is possible that the concepts of direct and opposite motions as defined above are not **well-defined !**

In order for the concept to be well-defined, we would have to show that no matter how a motion is written as a product of reflections, the parity (odd or even -ness) of the number of reflections used is always the same.

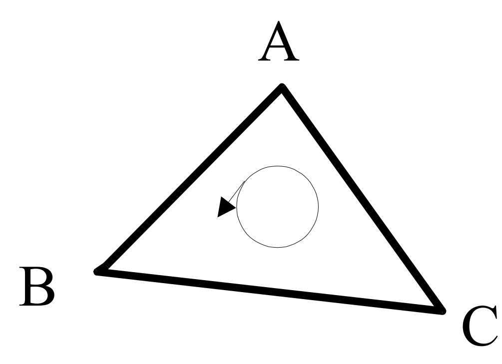
Direct and Opposite Motions

A better definition involves the concept of ***orientation*** which we will only deal with intuitively.

Given a triangle ABC in the plane, there are two possible orientations we can give to its vertices, denoted (ABC) or (ACB). We say that an orientation is ***positive*** if when transversing the sides of the triangle in the order of the vertices, the interior of the triangle is always on the left. The orientation is ***negative*** if the interior of the triangle is on the right. If (ABC) is a positive orientation, then (ACB) is a negative orientation.



Negative Orientation (ACB)



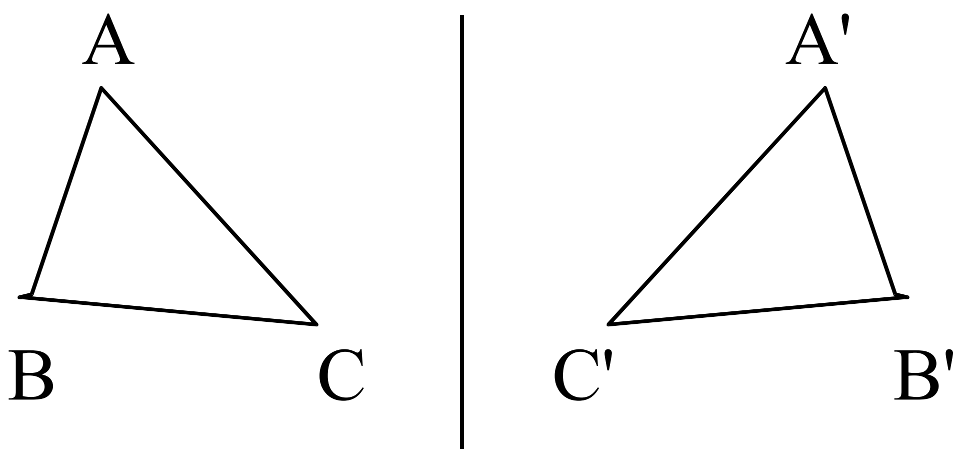
Positive Orientation (ABC)

Direct and Opposite Motions

Better Definition: A ***direct*** motion is a plane motion such that the orientation of each triangle is preserved, i.e. (ABC) = (A'B'C').

An ***opposite*** motion is a plane motion such that the orientation of each triangle is reversed, i.e. (ABC) = -(A'B'C').

Translations and rotations (the rigid motions) are direct motions while reflections and glide reflections are opposite motions.



Some Theorems

Theorem 2.4: A motion of the plane is uniquely determined by an isometry of one triangle onto another.

Theorem 2.5: The 4 Euclidean motions of the plane constitute a group of transformations.

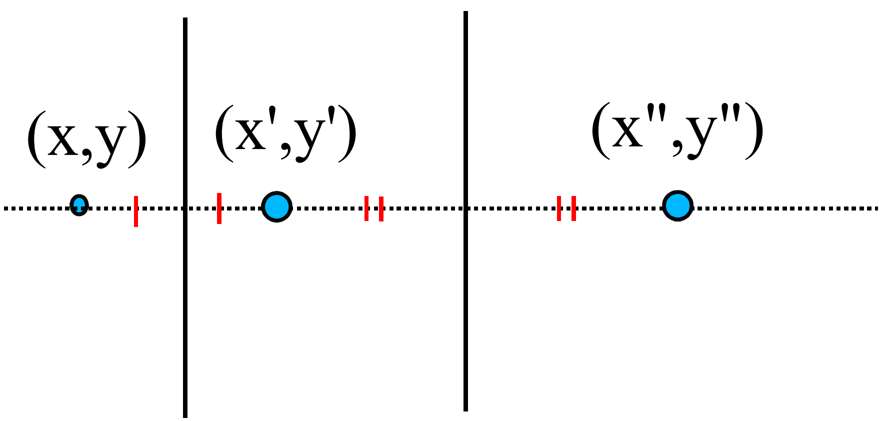
Theorem 2.6: If a transformation is a plane motion, then it is the product of three or fewer reflections. The converse is also true.

Theorem 2.7: Each finite group of isometries is either a cyclic group or a dihedral group.

Proof of Theorem 2.6

Proposition 1: The composition of two reflections with parallel axes of reflection is a translation in direction perpendicular to the axes and of magnitude equal to twice the distance between the axes.

Pf:

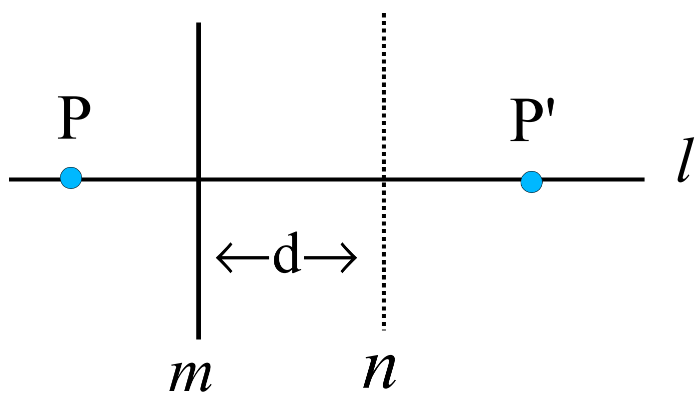


(x,y) $\rightarrow$ (x",y") is a translation

Proof of Theorem 2.6

Proposition 2: Any translation can be written (in many ways) as the composition of two reflections.

Pf: Let τ be a translation mapping point P to P'. Let *l* be the line joining P and P' and suppose the distance between P and P' is 2d.



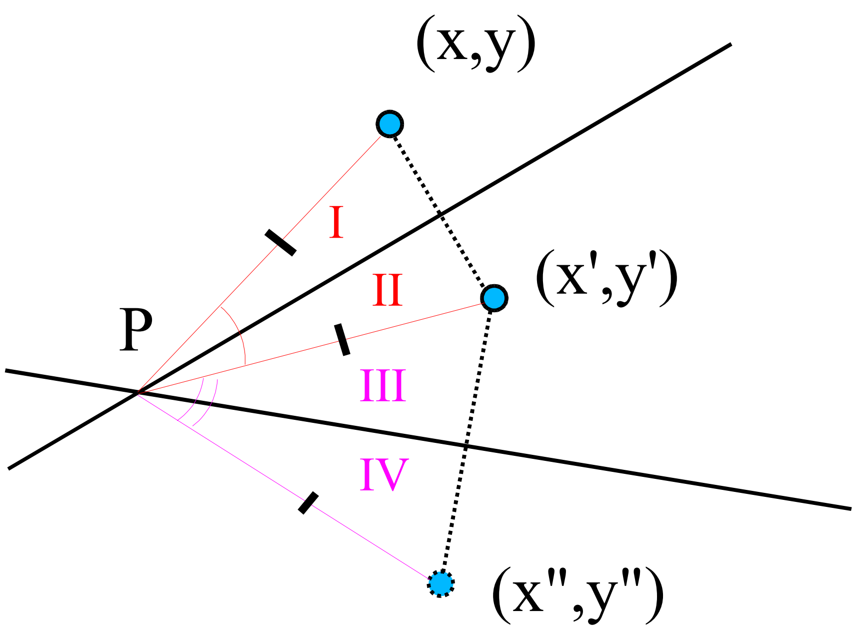
Arbitrarily, pick a line *m* perpendicular to *l*. Construct line *n*, parallel to *m* at distance d in the direction of vector PP'.

τ is the product of the reflection about *m* followed by the reflection about *n*.

Proof of Theorem 2.6

Proposition 3: The composition of two reflections with intersecting axes (meeting at point P with an angle of measure α) is a rotation of angle 2α with center P.

Pf:



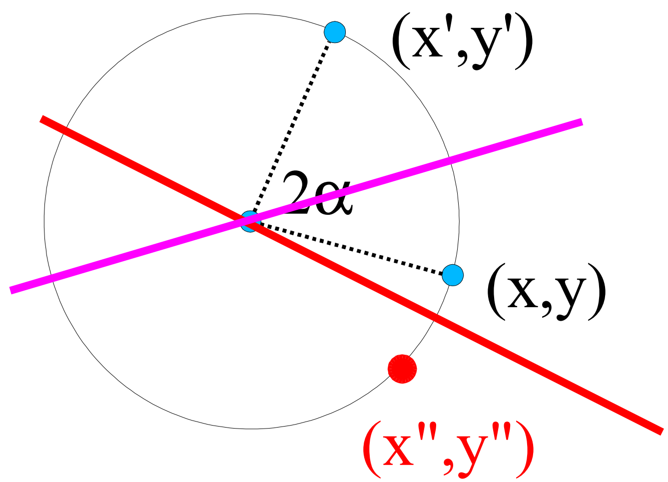
$\Delta I \cong \Delta II$ (SAS)

$\Delta III \cong \Delta IV$ (SAS)

Proof of Theorem 2.6

Proposition 4: Any rotation can be written (in many ways) as the composition of two reflections.

Pf:



Draw any line through the center, this is the axis of the first reflection.

Rotate this line through the center by an angle of α to obtain the axis of the second reflection.

Proof of Theorem 2.6

Proposition 5: The composition of 2 rotations with different centers is a translation or a rotation.

Pf: Let the first rotation have center O and angle α, and the second have center O' and angle β. Using Proposition 4, we can write the first rotation as the product of two reflections where the second reflection is about the line OO'. Again using Proposition 4, we can write the second rotation as the product of two reflections where the first reflection is about the line OO'. When we compose the two rotations, we write it as the product of 4 reflections, but the middle two are the same (and so, cancel). Thus, we get the composition as a product of 2 reflections. If the axes of these two reflections that are left are parallel, the result is a translation (happens when $\alpha+\beta= 2\pi$) otherwise it is a rotation. ┌

Proof of Theorem 2.6

Proposition 6: The composition of a rotation and a translation is a rotation.

Pf: Let the rotation have center O and suppose the translation is $\tau_{a,b}$. Now using the line through O which is perpendicular to the direction of vector (a,b) as the second reflection, write the rotation as the product of two reflections. Using this same line as the first reflection, write $\tau_{a,b}$ as a product of two reflections. The composition of the rotation and translation is thus the product of 4 reflections with the middle two being the same! This simplifies to a product of two reflections. The lines of these reflections can not be parallel, so the product is a rotation. ┌

Proof of Theorem 2.6

We have, with all the previous propositions, proved that the set of "rigid motions" form a group, a subgroup of the group of all motions. We now continue ...

Proposition 7: The composition of 3 reflections whose axes are parallel or concurrent is a reflection.

Pf: Case I : *Parallel axes*.

The product of the first two reflections is a translation. Rewrite this translation as a product of 2 reflections where the second has axis which is the same as the original third. In the rewritten product of 3 reflections, the last two cancel leaving a single translation.

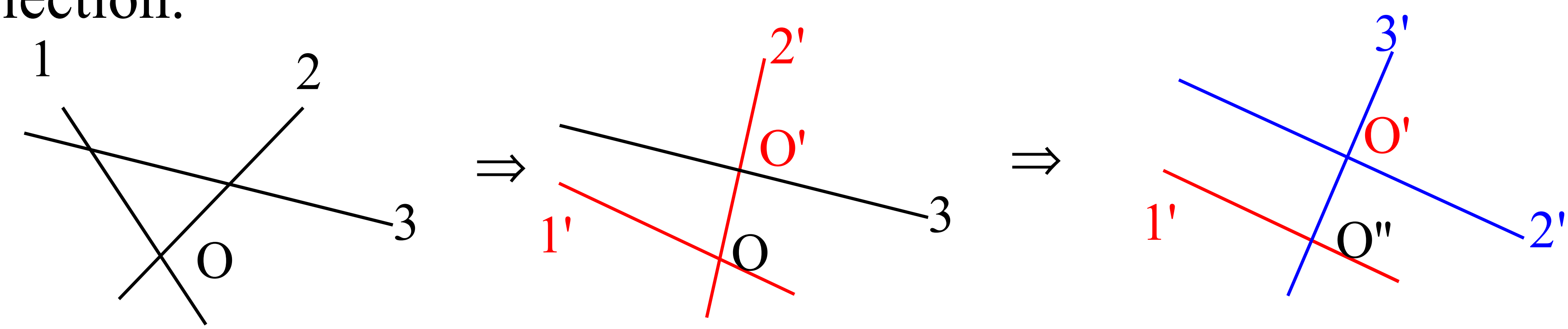
Case II: *Concurrent axes*.

Same argument as above, except that the product of the first two reflections is a rotation.

Proof of Theorem 2.6

Proposition 8: The composition of 3 reflections in all other cases is a glide reflection.

Pf: **Case I**: *First two reflections have axes which meet at point O.* The product of these two reflections is a rotation. Replace this rotation by a product of two reflections, the second of which has an axis perpendicular to the axis of the third reflection. The new second and original third reflections meet at a point O' so they form a rotation which we rewrite as a product of two reflections, the first of which we take to be parallel to the axis of the "new" first reflection.

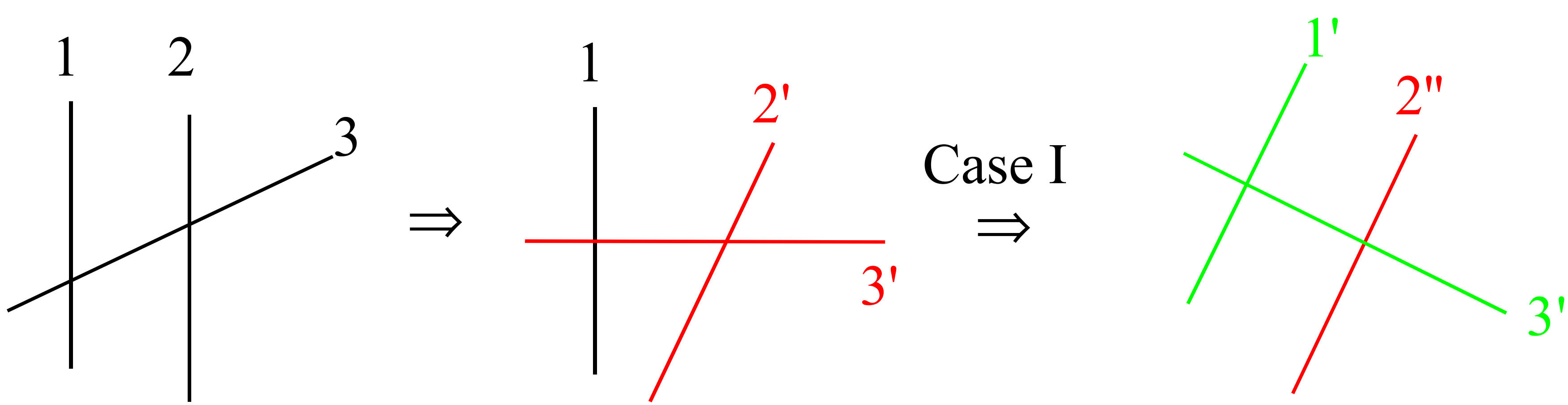


Proof of Theorem 2.6

Proof of Prop. 6 continued:

Case II: *First two reflections have parallel axes.*

The third reflection's axis is not parallel to the other two, so the product of the last two reflections is a rotation. Replace this rotation with a product of reflections, the axis of the first not parallel to the axis of the first reflection. Now, the problem has been reduced to the first case and we proceed as in that case.



Proof of Theorem 2.6

Finally,

in either case we have the first two reflections with parallel axes and so their product is a translation. The direction of this translation is parallel to the axis of the third reflection, thus the product of the three reflections is a glide reflection. ┌

Theorem 2.6

Every plane motion is the product of 3 or fewer reflections.

Pf: As we have seen, every translation or rotation is the product of 2 reflections, a reflection is just 1 reflection and a glide reflection is the product of 3 reflections. ⌜

Every composition of reflections is a plane motion.

Pf: Any composition of an even number of reflections is a translation or a rotation (in pairs they are and the rigid motions form a group), and can be replaced by a pair of reflections. So any number of reflections can be replaced by at most 3 reflections. If 0 or 2 we get a rotation or translation. If 1 or 3 we get a reflection or glide reflection. ⌜

Cyclic Groups

Consider the group Z_6 (the integers modulo 6) and the element 5 in this group. Notice that by adding 5 to itself enough times, we get all the elements of the group. 5, 5+5 = 4, 5+5+5 = 3, 5+5+5+5 = 2, 5+5+5+5+5 = 1, and 5+5+5+5+5+5 = 0. An element with this property in a group is called a ***generator*** of the group.

A group need not contain a generator. Any group which has a generator is called a ***cyclic*** group. A cyclic group can have more than one generator. Not every element of a cyclic group needs to be a generator. (In the cyclic group Z_6, 2 is not a generator since 2, 2+2 = 4, 2+2+2 = 0, 2+2+2+2 = 2, ... and we only obtain the elements 0,2 and 4.)

All cyclic groups are abelian.

Theorem 2.7

(**Leonardo daVinci**) *Every finite group of isometries of the Euclidean plane is either cyclic or dihedral.*

Pf: 1) A finite group of isometries can only contain rotations and reflections. Otherwise it would contain a translation, and a translation will lead to an infinite subgroup.

2) If the group only contains rotations, then they must have the same center, otherwise the group would contain translations. Since the group is finite, all the rotations are multiples of the smallest positive angle of rotation and the group is cyclic.

3) If the group contains at least one reflection then it is a dihedral group. The rotations form a subgroup which is cyclic. The # reflections = #rotations (a rotation composed with a fixed reflection will give all reflections as the rotation varies.)