

Combinatorial Proofs

Two Counting Principles

Some proofs concerning finite sets involve counting the number of elements of the sets, so we will look at the basics of counting.

Addition Principle: If A and B are disjoint finite sets with $|A|=n$ and $|B|=m$, then $|A \cup B| = n + m$.

Multiplication Principle: If a procedure can be broken into m stages and if there are r outcomes in the first stage, s outcomes in the second stage independent of the choice in the first stage, t outcomes in the third stage again independent of the previous choices then the total procedure has $rst\dots$ composite outcomes.

Examples

If I teach two classes and one class has 30 students and the other has 12 students, how many students do I teach?

There is not enough information to answer the question!!!

If there is no overlap, then the addition principle gives 42, but if there is an overlap the addition principle does not apply.

However, if we know that there are 2 students in both classes then we can use the addition principle as follows:

$$\begin{aligned} &\# \text{in first class only} + \# \text{in second class only} + \# \text{in both} \\ &28 + 10 + 2 = 40. \end{aligned}$$

Examples

Toss two dice. How many ways can the outcome not be doubles?

Think of the toss as a process with two stages, the first stage is the toss of one die and the second is the toss of the other.

1st method: There are 6 outcomes for the first stage, and to not get doubles, there are 5 outcomes for the second stage. By the product rule there are 30 possibilities. [Note that the number of outcomes for the second stage does not depend on which outcome occurred in the first stage.]

2nd method: Total number of outcomes (including doubles) is, by the product rule $6 \times 6 = 36$. Of these, 6 are doubles, so the number that are not doubles is $36 - 6 = 30$.

Examples

There are 15 different apples and 10 different pears. How many ways are there for Jack to pick an apple or a pear and then for Jill to pick an apple and a pear?

There are two natural stages, Jack's pick followed by Jill's pick. Jack chooses one out of 25 fruit, so there are 25 outcomes in this stage. To calculate the number of outcomes for Jill's pick we must know what Jack picked:

If Jack picked an apple, then Jill has $14(10) = 140$ choices.

If Jack picked a pear, then Jill has $15(9) = 135$ choices.

The product rule does not apply directly, but we can use the addition rule to break up the problem into subproblems in which it does apply.

Jack picks an apple + Jack picks a pear

$$15(140) + 10(135) = 2100 + 1350 = 3450$$

Lists (ordered arrangements)

In general when creating lists, if there are n_1 selections for first place, n_2 for second, ... n_r for the r -th place (each selection independent of the preceding) then the number of lists that can be created equals $n_1 n_2 \dots n_r$ (product rule).

If the selections are taken from an n -set and *repeats* (replacement) *are allowed* then the number of lists is given by $n n n \dots n = \mathbf{n^r}$.

If *no repeats* are allowed then the number of lists is given by

$$\mathbf{P(n,r) = n(n-1)(n-2)\dots(n-r+1)}$$

In the special case of complete lists (*permutations*) we have

$$\mathbf{P(n,n) = n!}$$

Permutation Problems

How many ways are there to pick 2 successive cards from a standard deck of 52 such that:

- a. The first card is an Ace and the second is not a Queen?**
- b. The first is a spade and the second is not a Queen?**

a) We are creating a list of two things. There are 4 choices for the first item and $(51 - 4) = 47$ choices for the second, so there are

$4(47) = 188$ ways to do this.

b) This time there are 13 choices for the first item, but the number of choices for the second depends on what was chosen first. If the Queen of Spades was picked first, there are $(51 - 3) = 48$ choices for the second card. If a different spade had been selected then there are $51 - 4 = 47$ choices for the second card. So, we get

$1(48) + 12(47)$ ways to do this.

Permutation Problems

There are 50 cards numbered 1 to 50. Two different cards are chosen at random. How many ways can this be done? In how many of these picks will one card be twice the other?

In this case we are not forming a list (there is no first card of the two selected), but we can still use the list idea to solve the problem. If we were forming a list, the number of possible lists would be $50(49)$ since the cards are different. But, two cards A and B would appear twice in these lists, once as A,B and again as B,A. So, to answer the question we should divide by 2 to get

$$\frac{1}{2}(50)(49) = 25(49).$$

To answer the second question, we note that the two cards are now distinguished, there is a larger one and a smaller one. If we select the larger one first there are 25 choices (it must be even), and then the smaller card is determined, so there are no additional choices.

Permutation Problems

How many integers greater than 6600 have distinct digits, not including the digits 7,8,or 9?

Since only 7 digits are allowed, we can not have any integers with more than 7 decimal places. We also can not have any with less than 5 decimal places and still be greater than 6600.

We look at the cases of 5, 6 and 7 decimal places separately.

For a 5 place integer, there are only 6 choices for the first digit since we can't use 0 there. There are also 6 choices for the second digit since we can't use the digit used in the first place, but we are now allowed to use 0. The total number is thus $6(6)(5)(4)(3)$.

Similarly, for a 6 place number we get $6(6)(5)(4)(3)(2)$ and a 7 place number give $6(6)(5)(4)(3)(2)(1)$. So, our answer is:

$$6(6)(5)(4)(3) + 6(6)(5)(4)(3)(2) + 6(6)(5)(4)(3)(2)(1) = 10,800$$

Permutation problems

Use the prime decomposition to show that every square integer has an odd number of divisors (including 1 and itself), and every non-square integer has an even number of divisors.

Observe that the divisors of an integer are made up of the same primes that make up the integer. We can count the number of divisors (factors) by listing, for each prime divisor, the number of times that prime appears in the factor (including the possibility that it doesn't appear at all). Thus, looking at the prime decomposition of an integer n we can say:

$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ has $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ factors.

If n is a square, all the exponents are even, so the number of factors is a product of odd numbers and so is odd.

If n is not a square, then at least one exponent is odd, so the number of factors has an even integer divisor and is even.

r-sets of an n-set

Subsets of size r of a set of size n are called *combinations of n elements taken r at a time*. The number of such subsets is given by the binomial coefficient $C(n,r)$, also written as $\binom{n}{r}$ and read as "n choose r".

We determine a formula for $C(n,r)$ by using an obvious, but important counting principle:

**If you count the elements of a set in two different ways -
YOU GET THE SAME ANSWER!**

Duh!!

Combinations

Example: We have already counted the number of lists of r elements taken from a set of n elements, $P(n,r)$. Now we count these in a different way. First select the r elements (**this can be done in $C(n,r)$ ways**) then arrange the r elements in lists in all possible ways (**$P(r,r)$ ways to do this**). So, ...

$$P(n,r) = C(n,r) \times P(r,r)$$

$$\begin{aligned} C(n,r) &= \frac{P(n,r)}{P(r,r)} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \\ &= \frac{n(n-1)\cdots(n-r+1)(\mathbf{n-r})(\mathbf{n-r-1})\cdots(\mathbf{2})(\mathbf{1})}{r! (\mathbf{n-r})!} = \frac{n!}{r!(n-r)!} \end{aligned}$$

Combinations

Given a set of n elements, there is only one subset that has 0 elements, i.e., the empty set. So,

$$C(n,0) = 1 \quad \forall n \in \mathbb{N}.$$

Also, there is only one subset that contains n elements. Thus

$$C(n,n) = 1 \quad \forall n \in \mathbb{N}.$$

Our formula for computing $C(n,r)$ would, in these two cases, give:

$$\binom{n}{0} = \frac{n!}{0!n!} \quad \binom{n}{n} = \frac{n!}{n!0!}$$

In order to have these formulas make sense, we *must* define $0! = 1$.

It is also convenient to *define* $C(n,r) = 0$ if $r < 0$ or $r > n$.

Poker Hands

A poker hand consists of 5 cards from a deck of 52 cards.

A ***four of a kind*** hand has 4 cards of the same denomination.

There are $\binom{13}{1}\binom{48}{1} = 13(48) = 624$ four of a kind hands.

A ***full house*** consists of 3 cards of one denomination and 2 cards of another denomination.

There are $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13(4)(12)(6) = 3,744$ full house hands.

A ***flush*** consists of 5 cards of the same suit but not consecutive.

There are $\binom{4}{1}\left[\binom{13}{5} - 10\right] = 5,108$ flush hands.

Committee Selections

Suppose that there are 10 Republicans, 8 Democrats and 2 Independent legislators who are eligible for committee membership.

There are $C(20,5)$ 5 member committees that can be formed.

How many 5 member committees have at least 2 Republicans?

$$C(10,2) C(18,3)$$

How many 6 member committees have as many Republicans as Democrats?

$$C(10,2) C(8,2) C(2,2) + C(10,3) C(8,3)$$

How many 5 member committees have no more than 2 Republicans?

$$C(10,5) + C(10,1) C(10,4) + C(10,2) C(10,3)$$

The Binomial Theorem

A *binomial* is an algebraic expression with two terms, like $x + y$. When we multiply out the powers of a binomial we can call the result a *binomial expansion*.

Of course, multiplying out an expression is just a matter of using the distributive laws of arithmetic, $a(b+c) = ab + ac$ and $(a + b)c = ac + bc$. In the examples below, I have multiplied out a few binomial expansions, without combining like terms or even writing products as powers. To make things easier to see, I have used a different color for the terms in each factor, so that you can see where they came from.

The Binomial Theorem

$$(x+y)^2 = (x+y)(x+y) = (x+y)x + (x+y)y = xx + yx + xy + yy$$

$$(x+y)^3 = (x+y)(x+y)(x+y) = (xx + yx + xy + yy)x + (xx + yx + xy + yy)y =$$
$$= xxx + yxx + xyx + yyx + xxy + yxy + xyy + yyy$$

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y) =$$
$$= (xxx + yxx + xyx + yyx + xxy + yxy + xyy + yyy)(x+y) =$$
$$= xxxx + yxxx + xyyx + yyxx + xxyx + yxyx + xyyx + yyyx +$$
$$+ xxxy + yxxy + xyxy + yyxy + xxyy + yxyy + xyyy + yyyy$$

Binomial Theorem

Notice that each term of the binomial expansions is a product composed of one item of each color. This suggests another way to look at this process of multiplication. Think of each factor in the product as a box containing two items, an x and a y (the colors are used here to keep track of which box an x or a y came from). We form *one* of the terms in the binomial expansion by picking either an x or a y from each of the boxes and multiplying our selections together. Since there are 2 choices for what we can pick from each of the boxes, the total number of possible different selections is $2 \times 2 \times 2 \dots \times 2 = 2^n$. Now look at the examples again, and notice that the number of terms in the binomial expansions is exactly the total number of possibilities, i.e., for the cube there are 8 terms, for the 4th power there are 16 terms, etc. This means that every possible way of choosing one item from each box gives a term in the binomial expansion.

Binomial Theorem

$$\begin{aligned}
 (x+y)^4 = & \mathbf{x} \mathbf{x} \mathbf{x} \mathbf{x} + \mathbf{y} \mathbf{x} \mathbf{x} \mathbf{x} + \mathbf{x} \mathbf{y} \mathbf{x} \mathbf{x} + \mathbf{x} \mathbf{x} \mathbf{y} \mathbf{x} + \mathbf{x} \mathbf{x} \mathbf{x} \mathbf{y} \\
 & + \mathbf{y} \mathbf{y} \mathbf{x} \mathbf{x} + \mathbf{y} \mathbf{x} \mathbf{y} \mathbf{x} + \mathbf{x} \mathbf{y} \mathbf{y} \mathbf{x} + \mathbf{y} \mathbf{x} \mathbf{x} \mathbf{y} + \mathbf{x} \mathbf{y} \mathbf{x} \mathbf{y} + \mathbf{x} \mathbf{x} \mathbf{y} \mathbf{y} \\
 & + \mathbf{x} \mathbf{x} \mathbf{y} \mathbf{y} + \mathbf{y} \mathbf{y} \mathbf{y} \mathbf{x} + \mathbf{y} \mathbf{y} \mathbf{x} \mathbf{y} + \mathbf{y} \mathbf{x} \mathbf{y} \mathbf{y} + \mathbf{y} \mathbf{y} \mathbf{y} \mathbf{y}
 \end{aligned}$$

Binomial Theorem

We would like to be able to say exactly how many terms are in each group (this would be the coefficient of the term if it was written out in the normal way). We can do this by counting because we know how the terms are formed. To get a term with, say, 3 x's and 1 y, all we have to do is decide which box we are going to pick the y from and then pick the x's from all the other boxes. Thus, the number of terms of this type is the number of ways we can pick 1 (box) from a set of 4 (boxes). This is the number of subsets of size 1 in a set of size 4, i.e. $C(4,1)$. Looking at the group of terms having 2 x's and 2 y's, we know that we get them by picking 2 of the 4 boxes to take the y's from, so there are $C(4,2) = 6$ such terms. Now, consider the first and last terms of this example. In order to get xxxx, you don't pick any boxes to take y's from, and you can only do this in one way, i.e., $C(4,0) = 1$. Similarly, to get yyyy, you must pick all the boxes to take y's from, and again there is only one way to do this since $C(4,4) = 1$. These two statements are true for any number of boxes, since $C(n,0) = C(n,n) = 1$ for any n.

Binomial Theorem

We can now drop the colors and write things normally to get:

$$\begin{aligned}(x + y)^4 &= C(4,0)x^4 + C(4,1)x^3y + C(4,2)x^2y^2 + C(4,3)xy^3 + C(4,4)y^4 \\ &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4\end{aligned}$$

The general case can be written as:

$$(x + y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \dots + C(n,k)x^{n-k}y^k + \dots + C(n,n)y^n$$

Because the numbers $C(n,k)$ appear as the coefficients of the terms in a binomial expansion, they are called *binomial coefficients*.

Counting Proofs

Using the formula for $C(n,r)$ it is easy to see that $C(n, r) = C(n, n-r)$.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

However, it is also possible to see this relationship without resorting to the formulas. To every subset of size r in an n set, there corresponds a unique subset of size $n-r$, namely, the complement of the r set. Thus, the total number of size r subsets must equal the total number of size $n-r$ subsets.

Reasoning this way supplies a *combinatorial proof* as opposed to the *algebraic proof* of manipulating formulas.

Combinatorial Proofs

$$C(n,m) C(m,k) = C(n,k) C(n-k, m-k)$$

To give a combinatorial proof of this binomial identity, we need to find a counting problem for which one side or the other is the answer and then find another way to do the count.

Let S be a set with n elements. Consider the ordered pairs (M,K) where $M \subseteq S$ with m elements and $K \subseteq M$ with k elements. The number of these ordered pairs is, $C(n,m) C(m,k)$ using the product rule and first counting the number of choices for M and then the number of choices for K . Now we count the ordered pairs in a different way. First choose a K , this can be done in $C(n,k)$ ways, and next count the number of M 's that contain this K . Since we must add $m-k$ elements to K to get an M , and these are in the complement of K , of size $n-k$, we have $C(n-k, m-k)$ ways to do this. So,

$$C(n,m) C(m,k) = C(n,k) C(n-k, m-k)$$

since these represent two ways to count the same thing.

An old favorite

Thm: If a set S has n elements then its power set has 2^n elements.

Pf: We count the number of subsets by size. There are $C(n,0)$ of size 0, $C(n,1)$ of size 1, ..., $C(n,r)$ of size r , etc. Thus, the total number of subsets of S is the sum:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^n \binom{n}{i}.$$

On the other hand, the binomial theorem gives, by setting $x = y = 1$:

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$$
$$2^n = (1 + 1)^n = \sum_{i=0}^n \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^n \binom{n}{i}.$$

Combinatorial Proofs

$$C(n,r) = C(n-1,r-1) + C(n-1,r).$$

$C(n,r)$ counts the number of r -sets in an n -set. We now count these in a different way. Single out one element of the n -set and think of it as being colored red.

Every r -set either contains the red element or not, so

$$C(n,r) = \# \text{ r-sets with red element} + \# \text{ r-sets without red element.}$$

To get a subset without the red element, we must select r elements out of $n-1$ (i.e., those that are not the red element) and there are $C(n-1,r)$ ways to do this.

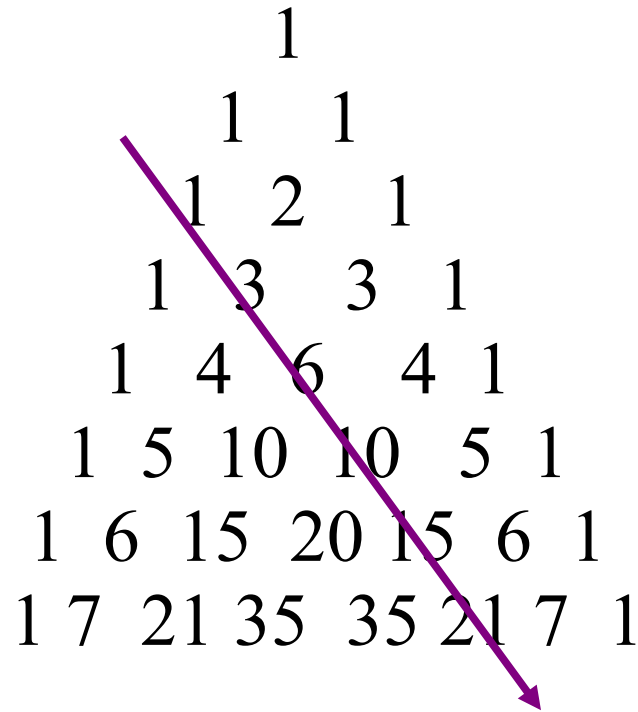
To get a subset with the red element, we must select $r-1$ additional elements from the $n-1$ non-red elements to add to the red element.

This can be done in $C(n-1, r-1)$ ways. So,

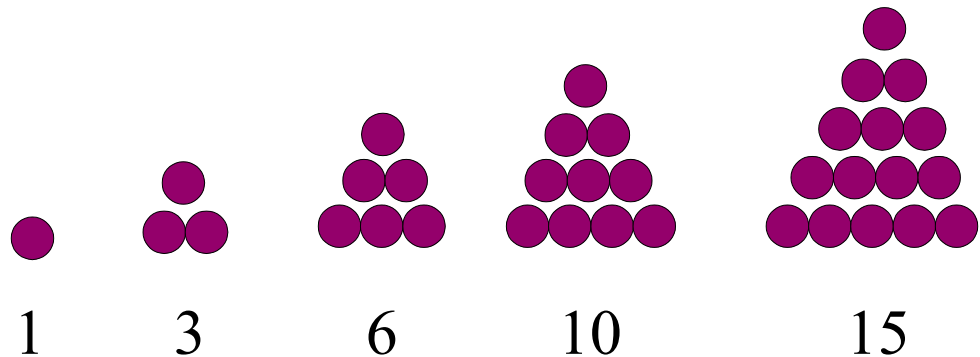
$$C(n, r) = C(n-1, r-1) + C(n-1, r).$$

Pascal's Triangle

Pascal's Triangle has many properties:

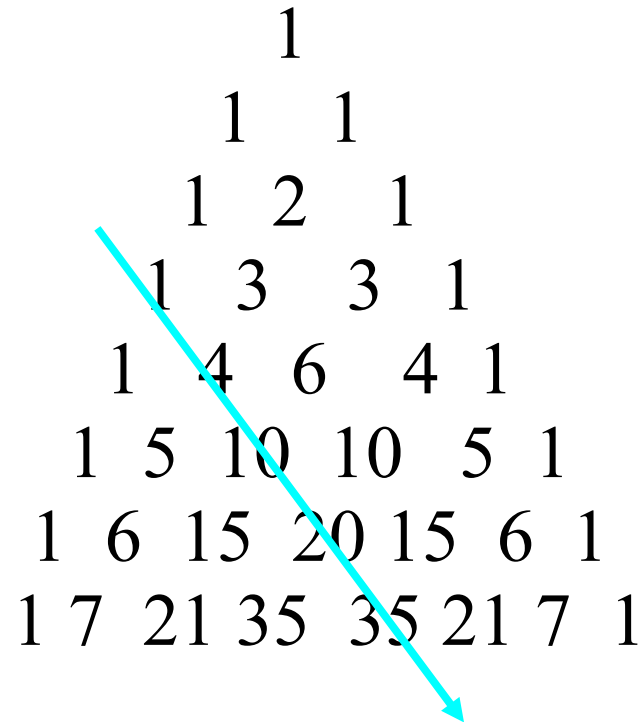


Triangular Numbers:

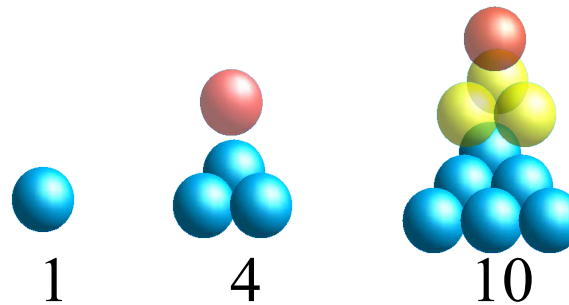


Pascal's Triangle

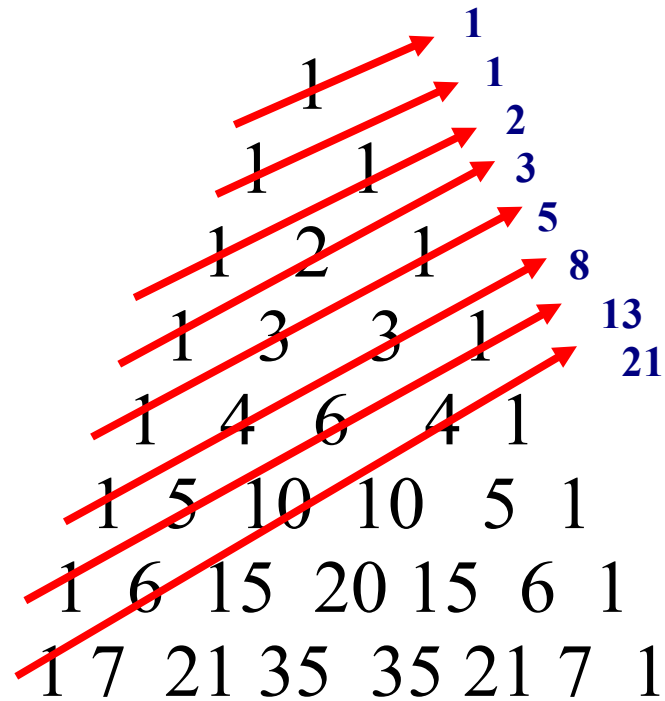
Pascal's Triangle has many properties:



Tetrahedral Numbers:

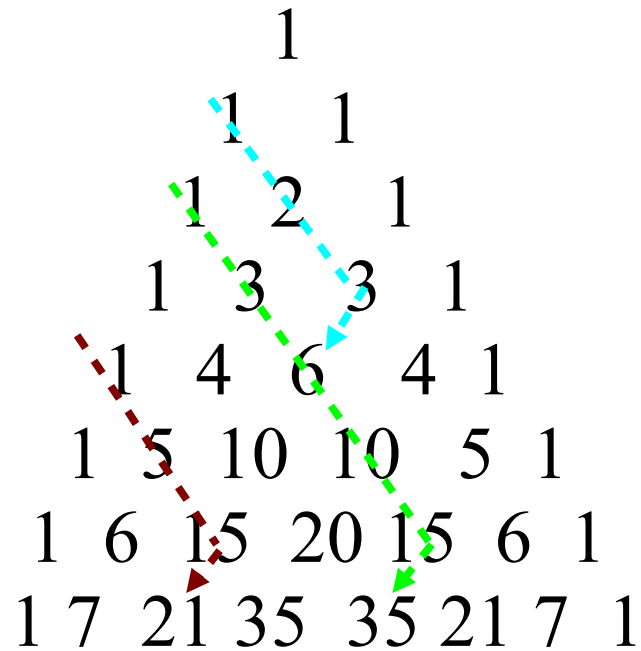


Pascal's Triangle



Fibonacci numbers hidden in the triangle

Pascal's Triangle



Illustrates the binomial identity:

$$\mathbf{C(n,0) + C(n+1,1) + \dots + C(n+k,k) = C(n+k+1,k)}$$

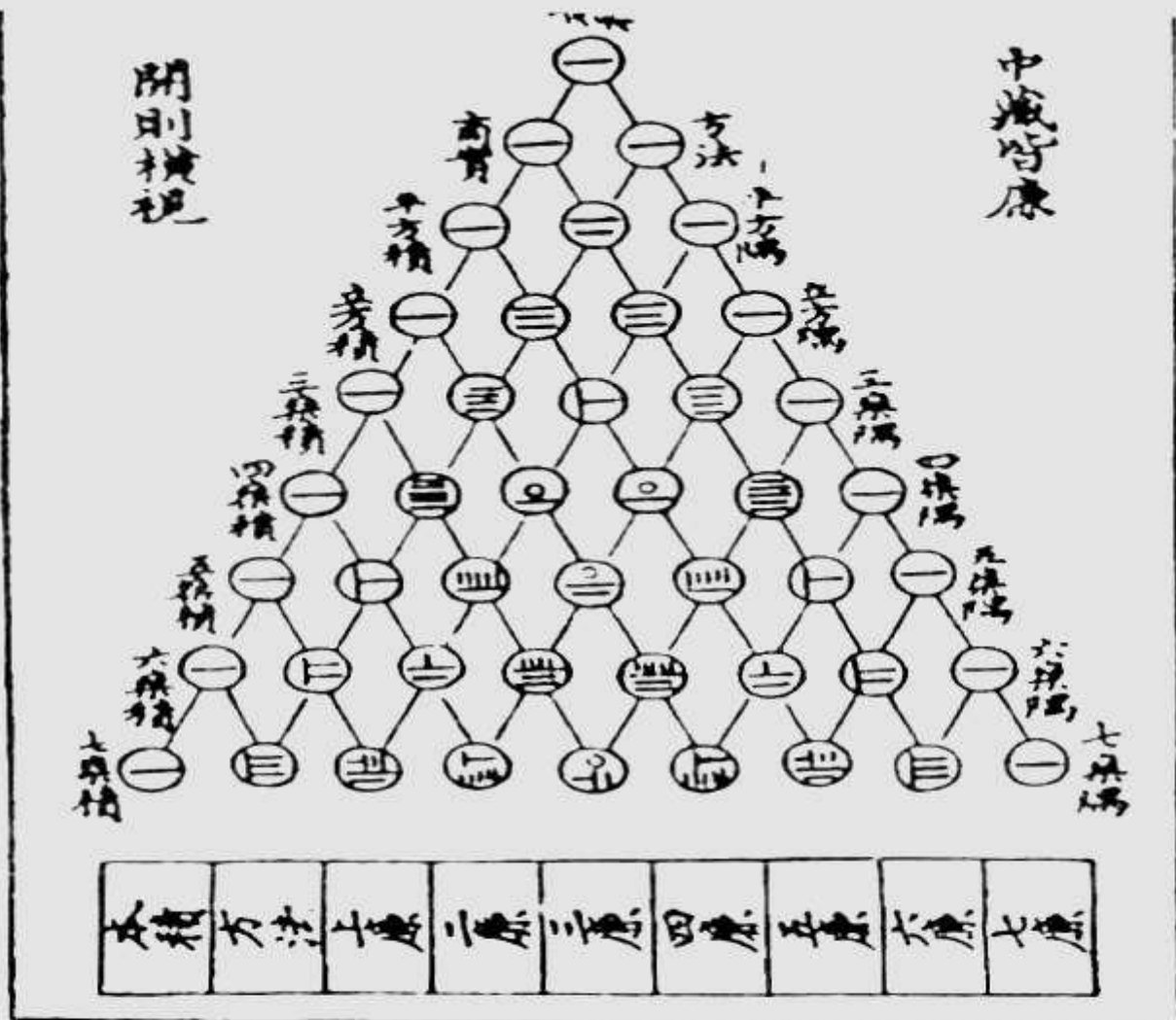
Pascal's Triangle



BLAISE PASCAL
(Brown Brothers)

We call the triangle Pascal's Triangle in honor of the French mathematician Blaise Pascal (1623 – 1662) who in 1653 wrote *Traité du triangle arithmétique* (**Treatise on the Arithmetic Triangle**). The properties we have mentioned, and many more, are collected in this work, many of them first discovered by Pascal.

HOWEVER, ...



“Pascal’s” Triangle shown in the 1303 text, *Szu-yuen Yu-chien* (The Precious Mirror of the Four Elements) by the Chinese mathematician Chu Shih-chieh.

Pascal's (?) Triangle

Even this is not the earliest reference we have to the triangle:

~**1100** Omar Khayyam mentions the triangle and does not claim credit for it. So he probably obtained it from earlier Chinese or Indian sources.

One More Counting Proof

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$$

Let a set S consist of n red balls and n blue balls. We want to count how many ways there are to select n balls (regardless of color) from S . The answer to this problem is clearly $C(2n, n)$. We now count in a different way.

Each selection of n balls contains a certain number, say x , of red balls and also $n-x$ blue balls. Clearly, x ranges from 0 to n . For any particular value of x , we can choose the red balls in $C(n, x)$ ways and the blue balls in $C(n, n-x)$ ways. So the number of ways to get an n set with x red balls is $C(n, x) C(n, n-x) = C(n, x)^2$ since we know that $C(n, x) = C(n, n-x)$. So, counting the n -sets by how many red balls they contain, gives:

$$C(2n, n) = C(n, 0)^2 + C(n, 1)^2 + \dots + C(n, n)^2$$