EULERIAN MOMENT EQUATIONS FOR 2-D STOCHASTIC IMMISCIBLE FLOW

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Abstract. We solve statistical moment differential equations (MDEs) for immiscible flow in porous media in the limit of zero capillary pressure, with application to secondary oil recovery. Closure is achieved by Taylor expansion of the fractional flow function and a perturbation argument. Previous results in 1-D are extended to 2-D. Comparison to Monte Carlo simulations (MCS) shows that the MDE approach gives a good approximation to total oil production. For such spatially integrated or averaged quantities MDEs may be substantially more efficient than MCS.

Key words. porous media, stochastic, random fields, moment equations

AMS subject classifications. 76T99, 76S05, 35L60, 35L65

1. Introduction.

Stochastic representations of subsurface geologic and flow properties have become commonplace due to the difficulty in complete and certain characterization of these properties. This leads to uncertainty in flow profiles in such porous media, so that statistical description of outcomes is appropriate. Additionally, from the perspective of practical macroscopic field-scale models, microscopic heterogeneities and flows can be viewed as random processes to be upscaled. In two-phase flow on a field scale, we are primarily interested in mean behavior and correlations between random fluctuations, which can be a measure of "macrodispersion" in certain upscaling contexts.

A "zeroth-order" model of mean flow with averages of geologic properties ignores these correlations. Monte Carlo simulations (MCS) of many realizations of geologic properties to estimate moments require much computation time and careful sampling techniques [14, 15, 37]. Macrodispersion theory in contaminant transport captures an approximate effect of fluctuations by modeling covariance functions in an equation for the mean concentration [7, 17]. This theory has a long history in subsurface contaminant transport and is closely related to eddy diffusion models of turbulence [20, p. 358 ff].

Rather than approximate covariance functions as macrodispersive terms, we solve PDEs for the covariance functions and the mean of saturation, extending previous results to 2-D. These *moment differential equations* (MDEs) are derived using a modified perturbation expansion. The MDEs allow direct approximation of the local mean and covariance functions for general boundary conditions and general, nonstationary stochastic geology [41].

1.1. Applications.

A statistical description of subsurface flow is of particular interest for secondary oil recovery. We apply our theory to a model of immiscible flow of oil and water phases that is commonly used for analytical studies in reservoir engineering [4], [11]. The principal difficulty in solving the model equation is a non-convex nonlinear flux function that leads to discontinuous solutions. The first

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main assumption is that capillary pressure (the difference in partial pressures of the two phases) is negligible. This is equivalent to assuming that advection is dominant: through a nonlinear diffusion term, the effect of capillary pressure is to smooth out sharp fronts. The relative importance of this diffusion term as well as the stability of such fronts depends on the ratio of viscosities of the two phases. We limit our example to a range of viscosity ratios that is consistent with stable, advection-dominated two-phase flow (see [11]). Second, we assume that the dependence of hydraulic conductivity $\mathbf{K}$ on (water) saturation s through relative permeabilities of the two phases can be neglected. This approximation has been commonly used in reservoir engineering, where it is often found to be valid in practice. The result is that the (water) saturation equation can be treated separately from equations for the velocity field and hydraulic head after solving the latter equations over steady boundary conditions (the velocity field is then steady as well).

The two main assumptions stated above greatly reduce the complexity of the problem and focus our efforts on understanding the impact of heterogeneity on a nonlinear advection equation. However, recent results have shown that the full extent of nonlinearity due to coupling of saturation and velocity equations cannot be neglected in some circumstances [3, 31]. In particular when the nonlinearity is strong (governed by the viscosity ratio) and heterogeneity is relatively weak, this nonlinearity becomes the dominant factor in dispersive mixing [16].

We further limit our example to 2-D horizontal flow. To fully explore the effects of heterogeneous reservoirs it is clear that 3-D flow cases must be considered. However, the key feature that differentiates 1-D flow from flow in higher dimensions is finite velocity-field correlation length in the latter. This difference is more easily first explored in a simple 2-D case. The moment equations extend to flow in three dimensions.

The pressure-saturation equations for 2-D horizontal flow of two incompressible immiscible fluids in porous media in the limit of vanishing capillary pressure and neglecting dependence of hydraulic conductivity on saturation are

$$\phi\,\mathbf{v}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\,\nabla h(\mathbf{x}), \qquad \nabla\cdot\mathbf{v}(\mathbf{x}) = 0, \tag{1.1}$$

$$\partial_t s(\mathbf{x},t) + \nabla\cdot[f(s(\mathbf{x},t))\mathbf{v}(\mathbf{x})] = 0. \tag{1.2}$$

These are considered valid from *laboratory* (centimeters) to *field* scales of reservoir depth (10–100 meters) and length (100–10,000 meters). Hydraulic conductivity $\mathbf{K}$ may be an anisotropic tensor; here, for simplicity, it will be an isotropic scalar K. Apply (1.1)–(1.2) to the flow of oil and water, for arbitrary fluid mobilities. Denote the *total velocity*, a scaled total volumetric flux of both fluids, by $\mathbf{v}$, hydraulic water head by h, and porosity (assumed constant) by ϕ. Under the assumptions stated above, the *fractional flow function* $f(s)$ represents the fraction of $\mathbf{v}$ due to water.

As is standard in subsurface applications, let $Y = \ln K$ be a random field with prescribed mean and covariance functions; e.g., it is often claimed that Y is multivariate Gaussian, based on empirical observations [12] (our method does not depend on such a limiting assumption). Through (1.1)–(1.2), $\mathbf{v}$ and s are thus random fields. No other underlying sources of uncertainty are considered in this study.

Under the assumptions stated above, with steady boundary conditions, a steady $\mathbf{v}$ can be determined from (1.1). We evolve s from the stochastic PDE (1.2), assuming known statistics of $\mathbf{v}$. We combine analytical and numerical techniques to model the propagation of uncertainty from an underlying random field $Y(\mathbf{x})$, through $\mathbf{v}(\mathbf{x})$ to the solution $s(\mathbf{x},t)$.

1.2. Previous work. Existing work on MDEs focuses mostly on advection equations with linear flux functions [18] and some nonlinear subsurface flow equations of a form different from (1.2) [2, 36, 39, 40, 44].

Langlo and Espedal [24, 25] present a macrodispersion approach for the stochastic version of (1.2). The flux function is expanded in a Taylor series, and high-order terms are neglected; then standard techniques lead to a representation of macrodispersivity as a function of flow velocity covariance. Dagan and Cvetkovic extended earlier Lagrangian streamlines theory from solute transport (linear advection) [6, 9] to immiscible two-phase flow [5, 8] (see also [32, 33] for a similar approach). This approach takes advantage of the steady velocity field, transforming 2-D flow to 1-D flow along streamlines. The authors formulate integral equations for moments from ensemble averages over the streamlines rather than a system of MDEs. Zhang, Tchelepi, and Li have further extended these results [42, 43]. Assumptions of steady, statistically homogeneous velocity fields appear to be required by this approach; however, it is possible that these restrictions may be relaxed.

An Eulerian MDE approach has been successful for single phase and multiphase pressure and velocity equations, and a natural next step is to extend the theory from flux equations to transport equations (as in [18]) and (1.2). The Eulerian framework differs from streamlines not only in formulation but also in that the MDEs need no velocity-distribution assumption and extension to transient velocity fields is relatively straightforward. The appeal of MDEs relative to macrodispersion approaches is that covariances are computed directly, so that it is not necessary to approximate them for inclusion in a mean equation. We have derived and solved second-order MDEs for (1.2) in 1-D, using three different perturbation approaches [21, 22]. One of these approaches is applied here in 2-D.

Equations are presented in §2. In this section we also discuss classification and the fundamental difference between MDEs in 1-D and MDEs in higher dimensions. In §3 we compare our numerical MDE solution to Monte Carlo simulations.

2. Moment Equations. We use a modified perturbation expansion, described in detail in [21], to derive statistical MDEs. Closure of the moment equations implicitly depends upon the assumption that random fluctuations in $Y = \ln K$ are small ($\sigma_Y \ll 1$). This fact theoretically limits our results to weak heterogeneity, although it has been found for stochastic models of tracer transport that similar results provide good approximations to moments in realistically heterogeneous domains. Moments of h and $\mathbf{v}$ are assumed known from (1.1) and moments of Y, and can be estimated from established theory (for example [38–41, 44] use MDEs).

The saturation equation (1.2), with initial data $s(\mathbf{x}, 0) = g(\mathbf{x})$, is

$$\partial_t s + \partial_{x_i}(f(s)v_i) = 0 \tag{2.1}$$

(Einstein summation convention assumed). We assume that g is known with certainty. Solutions are defined in terms of *vanishing-viscosity* as follows. Capillary pressure regularizes sharp fronts caused by the nonlinear advection term. To obtain a linear approximation to this effect, add $\epsilon_D \partial^2_{x_i} s$ with $\epsilon_D > 0$ to the right side of (2.1). Letting $\epsilon_D \to 0$ defines the vanishing-viscosity solution [34], which is the one we seek.

Examples of commonly used methods for deriving moment equations with applications to subsurface flow and transport can be found in sources cited previously, and additionally in [7, 13, 22, 28]. These methods originated in the correlation equations of turbulence models. Here we apply a modification to a standard asymptotic expansion, omitting details that can be found in the literature.

2.1. Deriving MDEs. Let $\langle\cdot\rangle$ denote the expectation operator, defined by

$$\langle\psi\rangle \equiv \int_\Omega \psi(\omega)\, dP(\omega) \tag{2.2}$$

for any integrable function $\psi : \Omega \to \mathbb{R}$ on the sample space Ω with probability measure P. We omit reference to ω in what follows. The random field Y is decomposed into deterministic mean plus random fluctuation: $Y = \langle Y\rangle + \delta Y$. Each field dependent on Y is represented by a formal power expansion in an as-yet unknown parameter ϵ:

$$h = \sum_{n=0}^{\infty} \epsilon^n h_n(\mathbf{x}), \quad \mathbf{v} = \sum_{n=0}^{\infty} \epsilon^n \mathbf{v}_n(\mathbf{x}), \quad s = \sum_{n=0}^{\infty} \epsilon^n s_n(\mathbf{x}, t). \tag{2.3}$$

The expansion parameter $\epsilon = \sigma_Y$ is shown to be appropriate within the context of the velocity and head equations (1.1) [7, pp. 184–190], [41]. For example, for single-phase, stationary uniform mean flow on an infinite domain in 1-D, v_0 is a deterministic scalar and σ_v^2 is shown to be approximated by $\epsilon^2 \langle v_1^2\rangle = v_0^2\sigma_Y^2$. Thus it is clear that our approach relies on an assumption of weak heterogeneity ($\sigma_Y \ll 1$). In 2-D, we denote the components of $\mathbf{v}_n$ by $(v_i)_n$ for $i = 1, 2$. Note that in the following, the term "order" applies to the power of ϵ rather than the order of the statistical moment.

Recall that we only need the decompositions of $\mathbf{v}$ and s here. The fractional flow function is expanded in a Taylor series around $\tilde{s} = \langle s_0\rangle + \epsilon\langle s_1\rangle + \epsilon^2\langle s_2\rangle$ (a second-order approximation to the mean saturation):

$$f(s) = f(\tilde{s}) + f'(\tilde{s})(s-\tilde{s}) + \frac{1}{2}f''(\tilde{s})(s-\tilde{s})^2 + \cdots, \tag{2.4}$$

where

$$s - \tilde{s} = \delta s_0 + \epsilon\delta s_1 + \epsilon^2\delta s_2 + \sum_{n=3}^{\infty}\epsilon^n s_n(\mathbf{x}, t).$$

The expansion center $\tilde{s}$ is our approximation to the mean saturation, so we retain $\tilde{s}$ as the argument of f in equations for each s_n, $n = 0, 1, 2$. This choice means that we have an inconsistency in order: we retain a higher-order argument ($\tilde{s}$) in what would otherwise be lower-order equations. We offer the following justification for this apparent asymptotic heresy. We found in 1-D analytical results that the standard asymptotic expansion leads to secular terms in the saturation moments (terms that grow in time and thus violate both physical bounds on saturation and the assumed asymptoticity of the expansion) [21, 23]. To control this secular behavior and to allow for combining the equations for $\langle s_0\rangle$, $\langle s_1\rangle$, and $\langle s_2\rangle$ into a single equation for mean saturation, it was necessary to make this adjustment to the flux term. This is not especially surprising, because the accuracy of the approximation of $f(s)$ depends on the base-point for the expansion, and among the linear combinations of $\langle s_0\rangle$, $\langle s_1\rangle$, and $\langle s_2\rangle$, the second-order mean may be the most representative base-point available.

Define $\tilde{\mathbf{v}} = \mathbf{v}_0 + \epsilon^2\langle\mathbf{v}_2\rangle$, consistent with the second-order theory for single-phase flow. The independent variables are $\mathbf{x}$ and t except where noted, and let $\cdot|_\mathbf{y}$ denote the replacement of $\mathbf{x}$ by some $\mathbf{y}$ different from $\mathbf{x}$. It is convenient and useful to derive equations for the more general two-point covariances $\langle s_1(v_i)_1|_\mathbf{y}\rangle$ and $\langle s_1 s_1|_\mathbf{y}\rangle$ rather than for one-point covariances. Define $c_{sv_i}(\mathbf{x},\mathbf{y},t) = \epsilon^2\langle s_1(v_i)_1|_\mathbf{y}\rangle$, $c_s = \epsilon^2\langle s_1 s_1|_\mathbf{y}\rangle$, $c_{v_iv_j} = \epsilon^2\langle (v_i)_1(v_j)_1|_\mathbf{y}\rangle$, $\langle s\rangle|_\mathbf{y} = \langle s\rangle(\mathbf{y}, t)$, $\sigma_{sv_i}(\mathbf{x}, t) = c_{sv_i}(\mathbf{x},\mathbf{x},t)$, and

$\sigma_s^2(\mathbf{x},t) = c_s(\mathbf{x},\mathbf{x},t)$. Let a caret over a variable denote the mapping $\widehat{\psi}(\mathbf{x},\mathbf{y},t) = \psi(\mathbf{y},\mathbf{x},t)$ for any function ψ. After taking equations for $\langle s_0\rangle$, $\langle s_1\rangle$, and $\langle s_2\rangle$ and consolidating the equations for mean saturation $\tilde{s}$, we obtain the following equations for mean and two-point covariance functions:

$$\partial_t \tilde{s} + \partial_{x_i}\Big[f(\tilde{s})\tilde{v}_i + f'(\tilde{s})\sigma_{sv_i} + \frac{1}{2}f''(\tilde{s})(v_i)_0\,\sigma_s^2\Big] = 0, \tag{2.5a}$$

$$\partial_t c_{sv_j} + \partial_{x_i}\Big[f(\tilde{s})c_{v_iv_j} + f'(\tilde{s})(v_i)_0\, c_{sv_j}\Big] = 0, \tag{2.5b}$$

$$\partial_t c_s + \partial_{x_i}\Big[f(\tilde{s})\widehat{c}_{sv_i} + f'(\tilde{s})(v_i)_0 c_s\Big] + \partial_{y_i}\Big[f\big(\widehat{\tilde{s}}\,\big)c_{sv_i} + f'\big(\widehat{\tilde{s}}\,\big)(v_i)_0|_{\mathbf{y}} c_s\Big] = 0; \tag{2.5c}$$

$$i,j = 1,2.$$

Recall that velocity moments $(v_i)_0$, $\tilde{v}_i$, and $c_{v_iv_j}$ are assumed known. The flux in (2.5a) consists of a nonlinear advective mean flux term and two covariance terms. In turbulence applications, terms such as c_{sv_j} often are referred to as transport by fluctuations [27]. Both (2.5b) and (2.5c) have advective flux terms and are coupled to the mean equation (2.5a) The approximation $\tilde{s}$ is (formally) second-order in σ_Y. However, due to the choice of $\tilde{s}$, higher-order effects are included in (2.5). We demonstrate this fact for the degenerate case $f(s) = s$, where the mean advective flux term in (2.5a) is $\partial_{x_i}[\tilde{s}\tilde{v}_i]$, or $\partial_{x_i}[(s_0 + \epsilon^2\langle s_2\rangle)((v_i)_0 + \epsilon^2\langle (v_i)_2\rangle)]$. The term $\epsilon^4\partial_{x_i}[\langle s_2\rangle\langle (v_i)_2\rangle]$ in this expression is a partial fourth-order correction. Again, the justification that we offer for this single higher-order correction in a lower-order equation is that it is needed to control secular behavior.

The concept of vanishing viscosity can be carried through to (2.5) also: first add the deterministic term $\epsilon_D\partial_{x_i}^2 s$ to (2.1) and carry out the expansion and derivation above. This adds diffusion terms $\epsilon_D\partial_{x_i}^2\tilde{s}$, $\epsilon_D\partial_{x_i}^2 c_{sv_j}$, and $\epsilon_D\partial_{x_i}^2 c_s$ to the right-hand sides of (2.5a), (2.5b), and (2.5c), respectively. Then find solutions in the limit $\epsilon_D \to 0$. In practice we solve the system (2.5) directly.

2.2. Classification. Classification is of interest for several reasons. Macrodispersion theory produces a parabolic equation for the mean, with a macrodispersivity that depends on $c_{v_iv_j}$ and $\langle s\rangle$. In contrast, we have shown that 1-D MDEs are hyperbolic [21, 22]. We expect the equations to be nearly hyperbolic since the original deterministic PDE (1.2) is hyperbolic. However, σ_{sv_i} and σ_s^2, rather than c_{sv_i} and c_s, appear in the mean equation. Therefore the special structure that allowed the reduction and classification of 1-D MDEs is not evident in 2-D MDEs, prohibiting classification of (2.5) as hyperbolic. Analysis and numerical solution both require that we append to (2.5) a redundant set of equations for $\widehat{\tilde{s}}$ and $\widehat{c}_{sv_i}$, making the expanded system symmetric under the permutation $\widehat{\psi}(\mathbf{x},\mathbf{y},t) = \psi(\mathbf{y},\mathbf{x},t)$, but not hyperbolic in general. To be consistent with (2.5), a numerical approximation of the expanded system must use initial conditions and evolution equations that obey the symmetry.

3. Solution. We numerically solve (2.5) with a first-order upwind scheme within the framework of CLAWPACK [26]. However, we do not take full advantage of the functionality of CLAWPACK due to the difficulty in defining a Riemann solver for the MDEs. The mean total oil production that we compute using MDEs is compared to MCS moment estimates.

3.1. MDEs. Our example problem is on a rectangular domain $R = [0,L_1]\times[0,L_2] = [0,2]\times[0,1]$, $\mathbf{x} = (x_1,x_2)\in R$, with $\partial_{x_2}h(\mathbf{x}) = 0$ on $x_2 = 0$ and $x_2 = 1$ and constant $h(\mathbf{x})$ at $x_1 = 0$ and $x_1 = 2$. The entire domain R is initially oil-saturated ($s = 0$), with water ($s = 1$) pumped in along $x_1 = 0$. We choose a form of the fractional flow function $f(s)$ that arises from quadratic relative permeabilities: $f(s) = s^2/(s^2+m(1-s)^2)$ (see [1]). Our method does not depend on any such specific choice of $f(s)$. We set the *viscosity ratio* m to 0.5 and porosity ϕ to 0.2. Statistical parameters for MDEs are $\langle Y \rangle = 0$ and $\sigma_Y^2 = 0.25$, and we use the exponential covariance function given by

$$C_Y(\mathbf{r}) = \sigma_Y^2 \, \exp\left[-\frac{|r_1|}{\lambda_1} - \frac{|r_2|}{\lambda_2}\right], \tag{3.1}$$

where $\mathbf{r} = (r_1, r_2)$. We set correlation lengths $\lambda_1 = \lambda_2 = 0.2$. This gives 5 correlation lengths in the x_2 direction and 10 correlation lengths in the x_1 direction.

The resulting MDE saturation moments for a typical case (Fig. 3.1 (c) and (d)) indicate a pair of saturation fronts moving at distinct speeds, with a rarefaction zone trailing the slow front, just as we have found previously in 1-D. The saturation variance is concentrated almost entirely in the region between the two fronts in the mean saturation, as might be expected. This non-physical bimodal behavior of the MDEs was shown rigorously in 1-D and 2-D for linear transport in stratified media in [10] and in 1-D for nonlinear advection equations similar to (2.5) in [22, 23]. The behavior is a consequence of the implicit truncation of the moment equations. (Although the modified expansion presented here differs slightly from the truncation approach used in [10] and [22], a similar truncation is implicit in our approach. In the nonlinear case, the resulting equations are nearly identical.) We studied this example to determine whether bimodality would be mitigated in 2-D for nonlinear advection with finite velocity-field correlation length. In spite of the negative implications of our result, we show in the next section that the mean oil production curve is well approximated by results from MDEs.

3.2. Comparison to MCS. Monte Carlo simulations are frequently used in applications to subsurface flow and transport and for reservoir modeling. The primary drawback is that MCS is computationally costly. Separate and very important issues include the problem of good coverage of the true sample space and the standard problem of autocorrelation in random number generators. However, it is a well-established method and serves as a standard for comparison.

To make the comparison of MDE to MCS as consistent as possible, both are computed on the ensemble velocity moments that are usually only computed for MCS. This comparison isolates the difference between the two methods at the level of computing saturation moments (kindly suggested by Neuman and Guadagnini [29]). Contours of MDE solutions using MCS velocity moments are shown in Fig. 3.1, where they are compared to MCS saturation moments.

The total oil produced is computed as the mass quantity that has exited the right boundary at time t. The volume of oil this represents is the same as the volume of water injected in this case, which is given by

$$P(t) \equiv \iint_A \phi \, s(\mathbf{x}, t) d\mathbf{A}, \tag{3.2}$$

(assuming unit domain length in the vertical and unit density of oil) in each simulation. Sample mean and variance of $P(t)$ are computed and compared to an estimate of mean production from MDE. The latter is obtained by replacing s in the integral with $\tilde{s}$. The results are shown in Fig. 3.2 using 200 simulations for MCS.

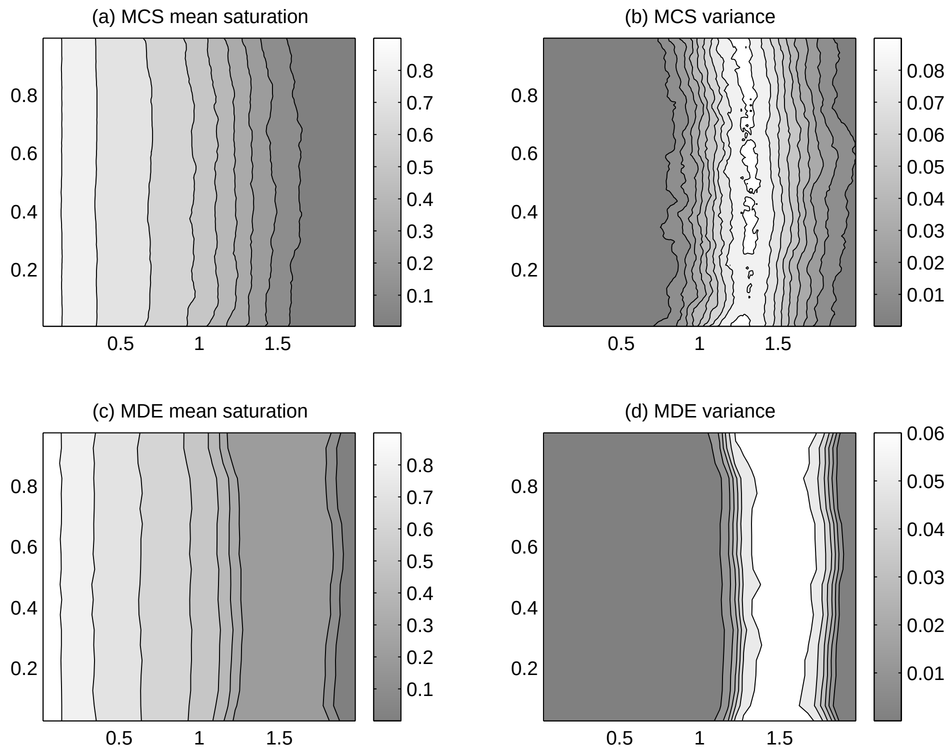

FIG. 3.1. *MCS saturation (a) mean and (b) variance and MDE (c) mean and (d) variance using MCS velocity moments.*

The MDE curve lies well within one sample standard deviation $\bar{\sigma}_s$ of the MCS curve, except near $t = 0$. The MDE curve has two sharp changes in slope, which are due to the passing of the two saturation fronts through the right boundary. We note again that the appearance of two saturation fronts rather than a diffused single front is a non-physical result. In each of the first two segments, the curve is nearly linear, which it should be for an intransient velocity field. Until water reaches the boundary (breakthrough), a constant flux of oil leaves the domain. The flux is again nearly constant between the saturation fronts, providing the second linear segment. Finally, after the second front passes the boundary, production decays smoothly as the rarefaction part of the mean saturation passes. The curves are fairly closely matched even with the erroneous bimodal sharp fronts in the mean saturation profile.

MCS provides solutions with high resolution; this is also a requirement, as the velocity field in each simulation may change significantly over short spatial scales. Averaging provides relatively smooth velocity and saturation moments. The MDE approach takes advantage of the smoothness of these moments before they are computed. Thus, we do not need a fine grid to capture detail. Grid refinement will certainly improve the solution to MDE, but refinement is not necessary to capture small-scale features.

On an SGI 300MHz MIPS12K processor with 256M memory 200 simulations on 200 by 100 grid nodes took about 8 hours. To achieve an error tolerance of $\delta = 10^{-2}$ at 97.5% confidence, we estimate that 4000 simulations are necessary, which

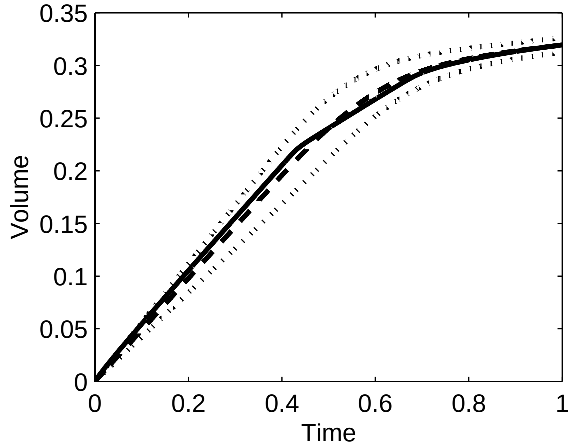

FIG. 3.2. *Total Production. Solid curve is obtained by MDE, dashed curve is MCS mean. One standard deviation is shown by dotted curves (200 realizations).*

would take about 160 hours on the same computer. This is much longer than the 5 hours it takes to obtain the solutions presented here using the MDE code. These are crude estimates; a careful comparison of costs for the two methods must allow for some attention to improving efficiency in both codes and should account for several differences between the methods, including the significantly higher storage cost for MDE. Additional details on the MCS approach are found in Appendix A and in [21].

4. Conclusions. The two-phase Eulerian MDEs presented here complement the traditional MCS approach. MDEs provide a first approximation to moments relatively quickly. Rather than compute several hundred sample saturation fields on fine grids and post-process to obtain moments, a single solution of MDEs is computed on a coarse grid. In spite of bimodality in MDE moments, we obtain a good match between MDE and MCS in the total oil production curve. For high accuracy, MCS is appropriate and, in real applications, it may be of most benefit to combine the two methods. It may also be possible to improve the accuracy of MDEs by retaining more terms in the perturbation expansions or using alternate methods such as cumulant expansions or semigroup perturbation expansions [30, 35].

The Eulerian MDE approach applies to any probability distribution of geologic and flow properties with any correlation functions. It depends solely on moments and does not require stationarity, thus removing a restriction that is common to many stochastic subsurface theories.

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Appendix A. MCS and convergence.

Monte Carlo simulations were performed by numerically solving the total velocity and saturation equations (1.1) and (1.2) over replicates of log hydraulic conductivity with correlation function (3.1). Replicates were generated with a Fast Fourier Transform algorithm developed by Gutjahr and colleagues [19]. Velocity equations were solved with a conjugate gradient algorithm. Saturation profiles were obtained using a first-order upwind approach just as for MDEs within the framework of CLAWPACK.

We performed simple tests for convergence of the mean and variance of saturation. A tolerance of 0.032 to within 97.5% confidence is estimated for results of the 350 simulations presented here. Cumulative average saturation from the first 150 simulations at representative grid points is shown in Fig. A.1. Although it is well-known that convergence of such estimates may appear imminent when in fact it is not, this is still a useful tool. The tolerance given above and the estimate of the number N_s of simulations needed to achieve the tolerance of $\delta = 10^{-2}$ given in §3.2 are based on the standard error formula

$$P\left[\frac{|\bar{s} - \langle s \rangle\,|}{\sigma_s / N_s^{1/2}} < 1.96\right] \approx 0.975$$

where $\bar{s}$ is the sample mean and both $\bar{s}$ and $\langle s \rangle$ are evaluated at a single point. For the error $|\bar{s} - \langle s \rangle\,|$ to be less than the tolerance we must have $N_s \geq (1.96\,\sigma_s/\delta)^2$ with σ_s replaced by the maximum sample standard deviation over the spatial grid.

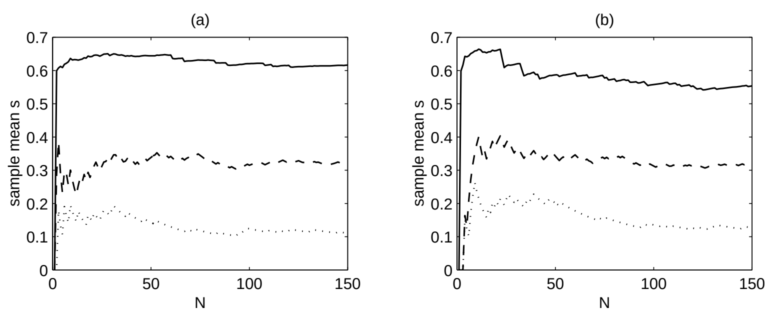

FIG. A.1. *Convergence of sample mean saturation at (a)* $x_2 = 0.4$ *and (b)* $x_2 = 0.6$*; in both,* $x_1 = 1.2$ *(solid),* $x_1 = 0.95$ *(dashed), and* $x_1 = 0.7$ *(dotted).*

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