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FINITE ELEMENTS AND FINITE DIFFERENCES: ARE THEY REALLY DIFFERENT, AND DOES IT MATTER?

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The finite-element method (FEM) and the finite-difference method (FDM) for approximating solutions of partial differential equations (PDE's) each have a large number of dedicated advocates. These individuals argue, with considerable justification, that their method is preferable for the problems of interest to them. Our experiences of the last several years have led us to a different perspective, which is that the two methods are actually quite similar, perhaps representing two different ways of viewing the same ideas. Indeed, it is possible to regard the FEM as a technique of deriving and justifying FD schemes that one would not have formulated otherwise, and FEM theory can illuminate FDM properties that are difficult to perceive within the customary FD framework. Worthwhile schemes can be developed that combine concepts traditionally associated with FEM's and FDM's.

We discuss some examples to illustrate our viewpoint. The modified method of characteristics (MMOC) for advection-dominated diffusion problems is formulated as a FEM and as a FDM, and we show that the natural advantages of the FEM version can be mimicked in the FDM setting. The block-centered FD scheme (BCFD), which is physically natural but inconsistent on nonuniform grids, can be proved to be convergent by FEM analysis. Other examples are cited in less detail, including FD proofs that yield FEM-like error estimates, the finite-volume-element (FVE) method, and analysis of local grid refinement.

1. Introduction

Having been trained in the analysis of FEM's and having subsequently worked in the petroleum industry on reservoir simulation, we have been exposed to widely differing views of numerical methods for PDE's. In structural mechanics, FEM's naturally reflect the minimization of energy, which is of great use in mathematical analysis of FEM's. When dealing with fluid flow, it is appropriate to define mass-conservation equations on subdomains, leading naturally to FDM's. The theory of FEM's is at home with "L^2" concepts such as energy and projection, while FDM's relate easily to "L^∞" ideas like maximum principles and monotonicity. The methods spring from different roots and represent very different ways of thinking.

This helps to explain why partisans of each method believe strongly in its superiority. Petroleum reservoir simulation began with FDM's over 30 years ago, before FEM's were widely known. The speed and simplicity of these methods relative to FEM's, together with their appealing physical interpretation of flows across faces of blocks, have entrenched them in the reservoir-simulation community. In fact, the standard upwind FD schemes are so attractive that they dominate the landscape to this day, despite the serious physical distortions that they introduce in many important problems [1]. Our experience tells us that alternative numerical schemes will not gain acceptance in reservoir simulation unless they can be posed in a FD framework. The main point of this paper is that, insofar as FEM's and FDM's are concerned, this is mainly a question of semantics. The methods are closely enough related that ideas inspired by one line of thought can make an impact in the other.

The remainder of this paper is outlined as follows. In Section 2, we describe a simple one-dimensional version of the MMOC for a linear advection-diffusion problem, indicating how some of its advantages accrue more naturally in the setting of FEM's than FDM's. At one time, we believed that this demonstrated a clear superiority of FEM's. However, we have since realized that the same advantages are available in the FD version, albeit in a manner requiring FEM thinking [2]. That is, the FEM concepts become a vehicle for the formulation of a FD scheme. Section 3 illustrates the past use of FEM ideas to analyze the BCFD method, finding that it converges even though the usual local truncation-error analysis would make one believe otherwise [1,3]. This is based on a strong analogy between BCFD and a mixed FEM, which could also aid in systematically formulating theoretically sound BCFD schemes on irregular geometries. We cite more recent work [4] that has used the ideas of finite-volume methods (FVM's), which in our view are members of the FDM family, to obtain FD error estimates of the same form as the FE estimates; this reinforces the FEM/FDM similarities. Finally, Section 4 sketches influences of FD ideas, through FVM's, on FEM's. One manifestation of this is the FVE method [5], in which the solution is represented by typical continuous FE trial functions, while the FD (or FV) concepts appear in the use of mass-conservation equations on associated cells. In a FE framework, this is equivalent to using discontinuous characteristic functions of the cells as test functions. Thus, here the FD approaches have led to a FEM, and theoretical analysis involves a combination of these viewpoints [6]. These ideas combine naturally with local-grid-refinement methods [7,8,9,10] to derive refinement techniques for mixed FEM's that can be analyzed with FEM machinery [11].

For the sake of brevity, the examples in this paper deal with simple one-dimensional problems. The descriptions could all be extended to more complicated situations in higher dimensions.

2. FE and FD Versions of MMOC

The MMOC, introduced by Douglas and the author [12] in the applied mathematics literature, is most simply exemplified in the context of a constant-coefficient linear one-dimensional advection-diffusion problem

$$u_t + vu_x - Du_{xx} = 0, \qquad x \in \mathcal{R},\ t > 0, \tag{1}$$

with appropriate initial data. This type of equation is difficult to solve with standard techniques when v/D is large compared to $1/L$, where L is the length scale of the problem.

The idea is to reduce (1) to a diffusion problem by using the characteristics of the associated hyperbolic equation

$$u_t + vu_x = 0 \tag{2}$$

to treat the advection. Accordingly, we let the role of "time" derivative be played by the total derivative (equal to zero in (2)), and a backward Euler approximation is

$$(u_t + vu_x)(x, t^n) \approx \frac{u^n(x) - u^{n-1}(x - v\Delta t)}{\Delta t}, \tag{3}$$

where superscripts represent time steps and $\Delta t = t^n - t^{n-1}$. Let

$$\hat{u}^{n-1}(x) = u^{n-1}(\hat{x}) = u^{n-1}(x - v\Delta t), \tag{4}$$

i.e., let $\hat{x}$ and $\hat{u}$ be x and u translated along characteristics.

To obtain a Galerkin FE version of the MMOC, confine the spatial domain to an appropriate bounded interval $[0, L]$, define the L^2 inner product $(f, g) = \int_0^L f(x)g(x)dx$, and choose a finite-dimensional test and trial subspace $\mathcal{M}$ of the Sobolev space $H^1(0, L)$. Typically, $\mathcal{M}$ consists of continuous piecewise polynomials with respect to a partition of $[0, L]$. Incorporate (3) in (1), multiply by a test function $\psi \in \mathcal{M}$, integrate over $[0, L]$, and integrate the diffusion term by parts. At time step t^n, the FE MMOC finds $U^n \in \mathcal{M}$ satisfying

$$(U^n, \psi) + \Delta t(DU_x^n, \psi_x) = (\hat{U}^{n-1}, \psi), \qquad \text{all } \psi \in \mathcal{M}. \tag{5}$$

In the FD case, assume for simplicity a uniform grid with points denoted by subscripts. The analogous centered-in-space equation is

$$U_i^n - \frac{D\Delta t}{(\Delta x)^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n) = \hat{U}_i^{n-1}, \qquad \text{all } i. \tag{6}$$

For nonlinear multidimensional problems, theory [13] and computational experience [14] demonstrate that these procedures can take longer accurate time steps than standard methods and can resolve steep moving fronts on coarser grids.

The issue that concerns us here is the evaluation of the right-hand sides of (5) and (6). In (6), a form of interpolation is necessary, since $\hat{x}_i$ may fall between grid points. To compare the two schemes, the appropriate choice of $\mathcal{M}$ is piecewise-linear polynomials, so linear interpolation would seem suitable. However, the FE version is second-order accurate in space, while the linearly interpolated FD scheme can be shown to be equivalent to an upwind method and is only first-order [12]; the greater accuracy of the FE procedure is borne out in computations [2]. One way to understand this is to see that the right-hand side of (5) is an integral that takes full account of the shape of $\hat{U}^{n-1}$, which is not in $\mathcal{M}$, while (6) only samples this function at grid points, causing numerical diffusion.

In the past, we have viewed this as an intrinsic advantage of the FE MMOC over the FD procedure. Closer examination reveals that this may not be so clear. If we replace $\hat{U}^{n-1}$ on the right-hand side of (5) by its L^2 projection $\tilde{U}^{n-1}$ into the trial space $\mathcal{M}$, the integrals against test functions from $\mathcal{M}$ in (5) are unchanged; that is, the FE MMOC cannot tell the difference between $\hat{U}^{n-1}$ and $\tilde{U}^{n-1}$. But $\tilde{U}^{n-1} \in \mathcal{M}$, so it is piecewise linear and is fully described by its values $\tilde{U}_i^{n-1}$ at the mesh points. Thus, the FD MMOC (6) with $\tilde{U}_i^{n-1}$ in place of $\hat{U}_i^{n-1}$ on the right-hand side would sample the old time level as accurately as the FE MMOC (5). Under the name of "least-squares interpolation," this was carried out in [2] and performed as expected. The name is suggested by the L^2 projection, which finds $\tilde{U}_i^{n-1}$ as a least-squares fit of $\hat{U}_i^{n-1}$. The rigorous calculation of $\tilde{U}_i^{n-1}$ is globally coupled, but we anticipate that this can be approximated with sufficient accuracy by a suitable local formula. It is unlikely that one would have developed this FD scheme without the FE viewpoint, but it may well give the MMOC access to applications such as reservoir simulation that are closely tied to FDM's.

3. FD Error Estimates

The BCFD method is derived naturally from FV ideas as follows. For the simple problem

$$-(a(x)u_x)_x = f(x), \qquad x \in I, \tag{7}$$

consider subintervals $[x_{i-1/2}, x_{i+1/2}]$ with midpoint x_i and length Δx_i. On a subinterval, the integral form of (7) is

$$-(au_x)(x_{i+1/2}) + (au_x)(x_{i-1/2}) = \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx, \tag{8}$$

i.e., the net flux equals the source. The most obvious FD scheme for (8) is

$$-a_{i+1/2}\frac{U_{i+1}-U_i}{\frac{1}{2}(\Delta x_i + \Delta x_{i+1})} + a_{i-1/2}\frac{U_i - U_{i-1}}{\frac{1}{2}(\Delta x_{i-1} + \Delta x_i)} = f_i \Delta x_i, \tag{9}$$

which, upon division by Δx_i, becomes the BCFD method for (7). Truncation-error analysis reveals that (9) is second-order correct on a uniform grid, first-order on a nonuniform grid if $\Delta x_i = (x_{i+1} - x_{i-1})/2$, and inconsistent if this condition does not hold.

When we first realized this, we attempted without success to produce a numerical example in which BCFD failed to converge to the correct solution. This lack of success was not accidental; subsequent investigations [1,3] showed that BCFD can be derived as a mixed FEM for which convergence theorems can be proved. Specifically, this mixed method approximates u and the flux au_x by discontinuous piecewise-constant and piecewise-linear polynomials, respectively, using the midpoint and trapezoidal quadrature rules in the corresponding variational equations. Details of the formulation appear in [1]. This may be viewed as dual to the well-known analogy between the usual point-centered FD scheme (which is at least first-order correct) and a Galerkin FEM. Based on our experience in the petroleum industry, we believe that this relationship between BCFD and mixed FEM's can be put to practical use in FD modeling of irregular geometries. In reservoir simulation, these occur due to geological phenomena such as faults, fractures, and pinchouts of layers. Some industry FD codes deal with such situations by defining logically rectangular grids that refer to the vertices of cells; the PDE approximations lack theoretical justification and can yield unsatisfactory results. An appropriate mixed FEM, formulated as a BCFD scheme, should circumvent this difficulty.

This further displays the utility of FEM ideas in developing and analyzing FD schemes. However, recent work of Samarskii, Lazarov, and Makarov [4] indicates that this may not be truly necessary, i.e., error estimates as strong as those for FEM's can be demonstrated by FD arguments. Using FV concepts such as those embodied simplistically in (8), they have obtained

$$||u - U||_{L^2} \leq C(\Delta x)^2 ||u||_{H^2}, \tag{10}$$

where the norms are discrete and C is independent of Δx and u. This points in the direction of saying that the long-held belief that FEM's have a greater order of accuracy and/or require less smoothness of the solution may not be as valid as previously thought.

4. FV Concepts in FEM's

Consider the FVE method for the simple model problem (7). The method uses the integral form (8) and seeks $U \in \mathcal{M}$, where $\mathcal{M}$ is a FE trial space, satisfying (8) for a finite set $\mathcal{J}$ of intervals of cardinality equal to the dimension of $\mathcal{M}$. The procedure has proved effective for a wide range of problems, particularly fluid flows [5,8]. Our purpose here is merely to see that FVE can be viewed as a FEM in which the test functions are the characteristic functions χ_J of the intervals J in $\mathcal{J}$. Posing such a FEM, we have

$$(-(au_x)_x, \chi_J) = (f, \chi_J). \tag{11}$$

Integrating by parts in (11), we see that the interior constancy of χ_J causes the interior integral to vanish, leaving only the endpoint terms and resulting in the form of (8); this is really just the divergence theorem applied to J. Further aspects of FVE and local grid refinement were noted in the introduction.

5. Conclusions

Returning to the question in the title, FEM's and FDM's are different, and the differences matter because each method allows a distinct set of useful concepts to be derived naturally. However, analogies and similarities abound, and each method has much to offer to devotees of the other, both theoretically and computationally. We see no reason to adhere strictly to one viewpoint, totally excluding the other. In particular, when dealing with practitioners accustomed to certain approaches, the skill of translating innovative ideas from one language to the other opens new opportunities; the practitioners may use techniques that they would be unwilling or unable to consider in the absence of the translation.

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