Infinite Sums, Infinite Products, and $\zeta(2k)$

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Contents

1 Review of Sequences and Series

The most basic concept is that of an infinite sequence (of real or complex numbers in these notes). For $p \in Z$, let $N_p = \{k \in Z : k \geq p\}$. An *infinite sequence of (complex) numbers* is a function $\underline{a} : N_p \rightarrow C$. Usually, for $n \in N_p$ we write $\underline{a}(n) = a_n$, and denote the sequence by $\underline{a} = \{a_n\}_{n=p}^{\infty}$. The sequence $\{a_n\}_{n=p}^{\infty}$ is said to converge to the limit $A \in C$ provided that for each $\epsilon > 0$ there is an $N_\epsilon \in Z$ such that $|a_n - A| < \epsilon$ for all $n \in Z$ with $n > N_\epsilon$. When this holds we write $lim_{n\rightarrow\infty} a_n = A$.

Def. A sequence $\{a_n\}_{n=p}^{\infty}$ of complex numbers is called a *Cauchy sequence* provided that for each $\epsilon > 0$ there is some $n_0 \in Z$ such that for $m, n \in Z, m \geq n_0$ and $n \geq n_0$ imply that $|a_m - a_n| < \epsilon$. When this is so, we write $lim_{m,n\rightarrow\infty} |a_m - a_n| = 0$.

Theorem 1.1 *(Cauchy; 1789 – 1857) A sequence of complex numbers cnverges if and only if it is a Cauchy sequence.*

Proof: Suppose $lim_{n\rightarrow\infty} a_n = A \in C$. Then given $\epsilon > 0$, there is an $n_0 \in Z$ for which $n \geq n_0$ implies $|A - a_n| < \epsilon/2$. Hence $m, n \geq n_0$ implies $|a_n - a_m| \leq |a_n - A| + |A - a_n| < \epsilon/2 + \epsilon/2 = \epsilon$. So $\{a_n\}$ is Cauchy.

For the converse we need to quote the famous Bolzano-Weierstrass theorem: Every bounded infinite set B of complex numbers has an accumulation point x, i.e., there is an $x \in C$ such that each punctured neighborhood $N_\delta(x) = \{z \in C : 0 \neq |x - z| < \delta\}$ contains a point of B (and hence contains infinitely many points of B).

Now suppose $\{a_n\}_{n=p}^{\infty}$ is a Cauchy sequence. For $\epsilon = 1$, there is an N_1 such that for all $m, n \geq N_1$, $|a_m - a_n| < 1$. Fix $n_0 \geq N_1$. Then for $m \geq n_0$, $|a_m| \leq |a_m - a_{n_0}| + |a_{n_0}| < 1 + |a_{n_0}|$. So if $\alpha = 1 + max\{|a_p|, |a_{p+1}|, \ldots, |a_{n_0}|\}$, we have $|a_m| < \alpha$ for all $m \geq p$. It is easy to show that if there are only finite many distinct values in $\{a_n\}_{n=p}^{\infty}$, then (because of the Cauchy condition) the values of a_n are constant for all sufficiently large n. And if $\{a_n\}_{n=p}^{\infty}$ has infinitely many values, then by the Bolzano-Weierstrass theorem there is an $A \in C$ such that each punctured neighborhood of A contains some term of $\{a_n\}_{n=p}^{\infty}$, and hence infinitely many.

Let $\epsilon > 0$ be given. Then there is an N_1 for which $m, n \geq N_1$ implies $|a_n - a_m| < \epsilon/2$. The punctured neighborhood $N_{\epsilon/2}(A)$ must have some point a_{n_0} with $n_0 \geq N_1$. Then $n \geq N_1$ implies $|a_n - A| \leq |a_n - a_{n_0}| + |a_{n_0} - A| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $lim_{n\to\infty} a_n = A$. ■

Using the Bolzano - Weierstrass theorem it is also easy to prove the following two results:

Theorem 1.2 *A bounded, nondecreasing sequence of real numbers converges to its least upper bound.*

Theorem 1.3 *Each bounded sequence of complex numbers has a convergent subsequence.*

Given the sequence $\{a_n\}_{n=1}^{\infty}$, we define the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums $s_n = a_1 + a_2 + \cdots + a_n$. The sequence $\{s_n\}_{n=1}^{\infty}$ is usually denoted by the "infinite sum" $\sum_{n=1}^{\infty} a_n$. By definition the series $\sum_{n=1}^{\infty} a_n$ *converges to the limit* S provided the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums converges to S. If we interpret Theorem 1.1 for the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums of the series $\sum_{n=1}^{\infty} a_n$, we have the following:

Theorem 1.4 *(Cauchy Criterion for Series). The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for each $\epsilon > 0$ there is some N_ϵ such that whenever $m \geq n > N_\epsilon$ we have $|a_n + a_{n+1} + \cdots a_m| < \epsilon$.*

Corollary 1.5 *If $\sum_{n=1}^{\infty} a_n$ converges, then* $\lim_{n\to\infty} a_n = 0$.

But the converse is false: In Calculus II the harmonic series $\sum_{n=1}^{\infty} 1/n$ is shown to diverge (i.e., *not converge*) even though $lim_{n\to\infty} \frac{1}{n} = 0$.

If $\sum_{n=1}^{\infty} a_n = A$, then $\sum_{n=N+1}^{\infty} a_n = A - \sum_{n=1}^{N} a_n$. Hence if $\left|\sum_{n=1}^{N} a_n - A\right| < \epsilon$, then $\left|\sum_{n=N+1}^{\infty} a_n\right| < \epsilon$.

The following limits are established for real values of z in Calculus II by using Taylor series. For complex non-real values of z they may be taken as definitions.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r| < 1. \tag{2}$$

$$sin\ z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \text{ for all } z \in C. \tag{3}$$

$$cos\ z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \text{ for all } z \in C. \tag{4}$$

$$log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}, \text{ if } |z| < 1. \tag{5}$$

$$-log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \text{ if } |z| < 1. \tag{6}$$

If $z = re^{i(\theta \pm 2n\pi)}$, $r > 0$, then $log(z) = ln(r) + i(\theta \pm 2n\pi)$, $-\pi < \theta \leq \pi$. One of the most fascinating equations in all of mathematics is due to Euler: $e^{i\theta} = cos\ \theta + i\ sin\ \theta$. It follows from Eqs. 1, 3 and 4.

For $n \geq 1$, put $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Then $S_n = \sum_{k=1}^{n} a_k = 1 - \frac{1}{n+1}$ converges to 1 as $n \to \infty$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Now consider the sequence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with partial sums $t_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2\cdot 2} + \cdots \frac{1}{n\cdot n} < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 + S_{n-1} < 2$. Since $\{t_n\}$ is strictly increasing and bounded above by 2, we have that $\{t_n\}_{n=1}^{\infty}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a good example for which it is not hard to see that it converges, but its sum is rather difficult to compute explicitly. In fact, one of the major goals of this chapter will have as a corollary the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{7}$$

Theorem 1.6 *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be complex sequences. If $\sum |a_n|$ converges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ converges.*

Proof: Choose $B \in R$ for which $|b_n| < B$ for all $n \geq 0$. Let $\epsilon > 0$ be given. Choose n_0 such that $q > p \geq n_0$ implies $\sum_{n=p+1}^{q} |a_n| < \epsilon/B$. Then $q > p \geq n_0$ implies $\left|\sum_{n=p+1}^{q} a_n b_n\right| \leq \sum_{n=p+1}^{q} |a_n b_n| \leq B \cdot \sum_{n=p+1}^{q} |a_n| < \epsilon$. We have used Cauchy's Criterion in both directions to complete the proof. ■

Corollary 1.7 *If $\sum_{n=0}^{\infty} a_n$ is a series of complex terms and $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof: Let $b_n = 1$ for all n and apply Theorem 1.6. ■

Def. A series $\sum a_n$ of complex terms is said to *converge absolutely* (or to be *absolutely convergent*) provided $\sum |a_n|$ converges. We note that Corollary 1.7 says that every absolutely convergent series is convergent. The converse is false. Recall from Calculus II that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Exercise 1.8 *(Comparison Test). If $\sum_{n=0}^{\infty} M_n$ is a convergent series of real numbers, and if $\sum_{n=0}^{\infty} a_n$ satisfies $|a_n| \leq M_n$ for all $n \geq p_0$ for some p_0, then $\sum_{n=0}^{\infty} a_n$ is (absolutely) convergent.*

Exercise 1.9 *If $\sum a_n$ converges, then $\sum c a_n = c \sum a_n$.*

Exercise 1.10 *If $\sum a_n$ and $\sum b_n$ converge, then $\sum (a_n + b_n) = \sum a_n + \sum b_n$.*

Theorem 1.11 *(Limit Comparison Test). Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers.*

(i) If $\limsup(a_n/b_n) < \infty$ and $\sum b_n < \infty$, then $\sum a_n < \infty$.

(ii) If $\liminf(a_n/b_n) > 0$ and $\sum b_n = \infty$, then $\sum a_n = \infty$.

Proof: (i) Choose β so that $\limsup(a_n/b_n) < \beta < \infty$. Then there exists an n_0 such that $a_n/b_n < \beta$ for all $n \geq n_0$. Hence $\sum_{n=n_0}^{\infty} a_n \leq \beta \cdot \sum_{n=n_0}^{\infty} b_n < \infty$.

(ii) Choose α satisfying $0 < \alpha < \liminf(a_n/b_n)$. Then for some n_0 we have $a_n/b_n > \alpha$ for all $n \geq n_0$. Hence $\sum_{n=n_0}^{\infty} a_n \geq \alpha \cdot \sum_{n=n_0}^{\infty} b_n = \infty$. ■

Theorem 1.12 *Suppose $a_{m,n} \in [0, \infty)$ for each $(m, n) \in \mathbf{N} \times \mathbf{N}$, and let $\phi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ be a bijection. With the understanding that a series of nonnegative real numbers converges to ∞ if it does not converge to a real number, and using the usual arithmetic for the symbol ∞, we have:*

(i) $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{k=1}^{\infty} a_{\phi(k)}$,

and

(ii) $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{m,n}\right)$.

Proof: First let α be any real number less than the right side of (i). Choose $k_0 \in \mathbf{N}$ such that $\alpha < \sum_{k=1}^{k_0} a_{\phi(k)}$. Next choose $m_0, n_0 \in \mathbf{N}$ such that $\{\phi(k) : 1 \leq k \leq k_0\} \subseteq \{(m, n) : 1 \leq m \leq m_0, 1 \leq n \leq n_0\}$. Then we have $\alpha < \sum_{k=1}^{k_0} a_{\phi(k)} \leq \sum_{m=1}^{m_0} \left(\sum_{n=1}^{n_0} a_{m,n}\right) \leq \sum_{m=1}^{m_0} \left(\sum_{n=1}^{\infty} a_{m,n}\right) \leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n}\right)$. Since α was arbitrary, $\sum_{k=1}^{\infty} a_{\phi(k)} \leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n}\right)$. Now let β be any real number less than the left side of (i), and chose $m_1 \in \mathbf{N}$ such that $\beta < \sum_{m=1}^{m_1} \left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{m_1} a_{m,n}\right)$ (use a generalization of Exercise 1.10). So we can choose $n_1 \in \mathbf{N}$ such that $\beta < \sum_{n=1}^{n_1} \left(\sum_{m=1}^{m_1} a_{m,n}\right)$. Now choose $k_1 \in \mathbf{N}$ such that $\{(m, n) : 1 \leq m \leq m_1, 1 \leq n \leq n_1\} \subseteq \{\phi(k) : 1 \leq k \leq k_1\}$. So $\sum_{n=1}^{n_1} \left(\sum_{m=1}^{m_1} a_{m,n}\right) = \sum_{m=1}^{m_1} \left(\sum_{n=1}^{n_1} a_{m,n}\right) \leq \sum_{k=1}^{k_1} a_{\phi(k)} \leq \sum_{k=1}^{\infty} a_{\phi(k)}$, i.e., $\beta < \sum_{k=1}^{\infty} a_{\phi(k)}$. As β was arbitrary, $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n}\right) \leq \sum_{k=1}^{\infty} a_{\phi(k)}$, and (i) is proved.

Equality (ii) will follow from (i) if we can show that $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{n,m}\right) = \sum_{k=1}^{\infty} a_{\phi(k)}$, since the left side of this is the right side of (ii) with the roles of m and n reversed. So write $b_{m,n} = a_{n,m}$; and $\psi(k) = (m, n)$ if $\phi(k) = (n, m)$. Then $b_{\psi(k)} = a_{\phi(k)}$ and $\psi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ is a bijection. Applying (i) to $b_{m,n}$ and ψ, we have $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{n,m}\right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} b_{m,n}\right) = \sum_{k=1}^{\infty} b_{\psi(k)} = \sum_{k=1}^{\infty} a_{\phi(k)}$. ■

Theorem 1.13 *(Main Rearrangement Theorem). Suppose $c_{m,n} \in C$ for each $(m, n) \in \mathbf{N} \times \mathbf{N}$ and $\phi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ is a bijection. If any of the three sums*

(i) $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |c_{m,n}|\right)$, $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} |c_{m,n}|\right)$, $\sum_{k=1}^{\infty} |c_{\phi(k)}|$ *is finite, then all of the series*

(ii) $\sum_{n=1}^{\infty} c_{m,n}, \;\; m = 1, 2, 3, \ldots$

(iii) $\sum_{m=1}^{\infty} c_{m,n}, \;\; n = 1, 2, 3, \ldots$

(iv) $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} c_{m,n}\right), \;\; \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} c_{m,n}\right), \;\; \sum_{k=1}^{\infty} c_{\phi(k)}$

are absolutely convergent, and the three series in (iv) all have the same sum.

Proof: By Theorem 1.12 the three series in (i) all have the same sum, which we are assuming to be finite. And since no term of a convergent series of nonnegative terms can be ∞, it follows that all series in (ii) and (iii) are (absolutely) convergent.

Write $\sum_{n=1}^{\infty} c_{m,n} = b_m, \; m = 1, 2, 3, \ldots$. Since $|b_m| = \lim_{q\to\infty} |\sum_{n=1}^{q} c_{m,n}| \leq \lim_{q\to\infty} \sum_{n=1}^{q} |c_{m,n}| = \sum_{n=1}^{\infty} |c_{m,n}|$ for all $m \in \mathbf{N}$, the Comparison Test (Exercise 1.8) yields $\sum_{m=1}^{\infty} |\sum_{n=1}^{\infty} c_{m,n}| = \sum_{m=1}^{\infty} |b_m| \leq \sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} |c_{m,n}|) < \infty$, so the first series in (iv) is absolutely convergent. By a similar argument the second series in (iv) is absolutely convergent. And by the first sentence of this proof so is the third. Write $\sum_{k=1}^{\infty} c_{\phi(k)} = s \in C$. We shall next show that $\sum_{m=1}^{\infty} b_m = s$, i.e., the first and third series in (iv) have the same sum. That the second and third have the same sum can be proved in a similar manner.

Let $\epsilon > 0$ be given. Choose $k_0 \in \mathbf{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |c_{\phi(k)}| < \epsilon/3, \tag{8}$$

and

$$\left| s - \sum_{k=1}^{k_0} c_{\phi(k)} \right| < \epsilon/3. \tag{9}$$

Next choose $p_0, q_0 \in \mathbf{N}$ such that $\{\phi(k) : 1 \leq k \leq k_0\} \subseteq \{(m,n) : 1 \leq m \leq p_0, \; 1 \leq n \leq q_0\}$. Then whenever $p \geq p_0$ and $q \geq q_0$, each term of the finite sum $\sum_{k=1}^{k_0} c_{\phi(k)}$ appears as a term in the finite sum $\sum_{m=1}^{p} (\sum_{n=1}^{q} c_{m,n})$. So subtracting those terms from the latter sum and using Eq. 8, we have

$$\left| \sum_{m=1}^{p} \left(\sum_{n=1}^{q} c_{m,n} \right) - \sum_{k=1}^{k_0} c_{\phi(k)} \right| \leq \sum_{k=k_0+1}^{\infty} |c_{\phi(k)}| < \epsilon/3. \tag{10}$$

We claim that

$$p \geq p_0 \text{ implies } \left| s - \sum_{m=1}^{p} b_m \right| < \epsilon. \tag{11}$$

(If true, $\sum_{m=1}^{\infty} b_m = s$ and the proof is complete.) So fix $p,\ p \geq p_0$. Since $\lim_{q\to\infty} \sum_{n=1}^{q} c_{m,n} = b_m$ $(m = 1, 2, 3, \ldots)$, it follows that
$\lim_{q\to\infty} \sum_{m=1}^{p} (\sum_{n=1}^{q} c_{m.n}) = \sum_{n=1}^{p} b_m$. Thus we may choose some $q,\ q \geq q_0$, such that

$$\left| \sum_{m=1}^{p} \left(\sum_{n=1}^{q} c_{m,n} \right) - \sum_{m=1}^{p} b_m \right| < \epsilon/3. \tag{12}$$

Then Eq. 11 follows from Eqs. 9, 10 and 12. ■

2 Infinite Products

Start with a sequence $\{u_n\}_{n=1}^{\infty}$ of complex numbers and form the partial products $\pi(p,q) = \prod_{n=p}^{q} u_n$. Suppose there is some $p \in \mathbf{N}$ for which $\pi_p = \lim_{q\to\infty} \pi(p,q) \in C$, and for which $u_n \neq 0$ for $n \geq p$. Then we say the **VALUE** of $\prod_{n=1}^{\infty} u_n$ is the number $\left(\prod_{n=1}^{p-1} u_n\right) \left(\lim_{q\to\infty} \prod_{n=p}^{q} u_n\right)$. (If $p = 1$, interpret $\prod_{n=1}^{p-1} u_n$ as 1.) If $\pi_p = \lim_{q\to\infty} \pi(p,q) \neq 0$,we say $\prod_{n=1}^{\infty} u_n$ *converges to* its value. If $u_n = 0$ for infinitely many n, $\prod_{n=1}^{\infty} u_n$, *diverges* but has value 0.

Note: Neither the convergence nor the value of $\prod_{n=1}^{\infty} u_n$ is affected by the paticular choice of p for which $u_n \neq 0$ where $n \geq p$.

All infinite products not covered by the above definition are said to *diverge*. If a divergent product has the value zero, we say it diverges to 0. This occurs if $u_n = 0$ for infinitely many n, or else $u_n \neq 0$ for $n \geq p$ but $lim_{q\to\infty} \prod_{n=1}^{q} u_n = 0$. A convergent product has the value 0 if and only if at least one but only finitely many of its factors equals 0.

Examples to illustrate the definitions

Example 1. $\prod_{n=1}^{\infty} \frac{n}{n+1}$ diverges to 0, since no factor equals 0 and $\prod_{n=1}^{q} \frac{n}{n+1} = \frac{1}{q+1} \to 0$.

Example 2. $\prod_{n=1}^{\infty} \frac{n^2-1}{n^2}$ converges to 0, since $u_1 = 0$, $u_n \neq 0$ for $n \geq 2$, and $\prod_{n=2}^{q} u_n = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdots \frac{(q-1)(q+1)}{2q} \to \frac{1}{2} \neq 0$.

Example 3. $\prod_{n=1}^{\infty} \frac{n-1}{n}$ diverges to 0. Here $u_1 = 0$, $u_n \neq 0$ for $n \geq 2$, but for any $p > 1$, $\prod_{n=p}^{q} \frac{n-1}{n} = \frac{p-1}{q} \to 0$ as $q \to \infty$.

Example 4. $\prod_{n=1}^{\infty}[1 + (-1)^n]$ diverges to 0.

Example 5 $\prod_{n=1}^{\infty}(-1)^n$ and $\prod_{n=1}^{\infty} \frac{n+1}{n}$ both diverge but not to 0.

Theorem 2.1 *(Cauchy Criterion) An infinite product* $\prod_{n=1}^{\infty} u_n$ *converges if and only if for each* $\epsilon > 0$ *there is an* $N \in \mathbf{N}$ *such that*

$$\left|1 - \prod_{n=r}^{s} u_n\right| < \epsilon \textit{ whenever } s \geq r \geq N. \tag{13}$$

Proof: Choose $p \in \mathbf{N}$ such that $u_n \neq 0$ for $n \geq p$. (This p is fixed throughout the entire proof.) Suppose the product converges and let $lim_{q\to\infty} \prod_{n=p}^{q} u_n = L \in C \setminus \{0\}$. Since $L \neq 0$, there is a $\delta > 0$ such that $\left|\prod_{n=p}^{q} u_n\right| > \delta$ for $q \geq p$. (To see this, first note that there is an N_1 for which $q \geq N_1$ implies $||\pi(p,q)| - |L|| < \frac{|L|}{2}$, and hence $\frac{|L|}{2} < |\pi(p,q)| < \frac{3|L|}{2}$. Put $\delta = \frac{1}{2}\left\{min\left\{|\pi(p,p)|, \ldots, |\pi(p,N_1)|, \frac{|L|}{2}\right\}\right\}$. So $0 < \delta < |\pi_{p,q}|$ for all $q \geq p$ as claimed, and also $\frac{1}{\pi(p,q)} < \frac{1}{\delta}$ for all $q \geq p$.) Let $\epsilon > 0$ be given. By Cauchy's Criterion for sequences applied to the sequence $\{\pi_{n=p}^{q} u_n\}_{q=p}^{\infty}$, there is an $N \in \mathbf{N}$, $N > p$, such that $\left|\prod_{n=p}^{r-1} u_n - \prod_{n=p}^{s} u_n\right| < \delta\epsilon$ whenever $s \geq r \geq N$. Multiply both sides by $\left|\prod_{n=p}^{r-1} u_n\right|^{-1}$ to obtain $|1 - \prod_{n=r}^{s} u_n| < \left|\prod_{n=p}^{r-1} u_n\right|^{-1} \delta\epsilon < \epsilon$ for $s \geq r \geq N$, which is Eq. 13.

Conversely, suppose Eq. 13 holds. Take $\epsilon = \frac{1}{2}$ and let N_0 be a corresponding N satisfying Eq. 13 with $M_0 > p$. Then we have

$$\frac{1}{2} < \left|\prod_{n=r}^{s} u_n\right| < \frac{3}{2}, \text{ for } s \geq r \geq N_0.$$

Put $r = N_0$, so

$$\frac{1}{2} < \left|\prod_{n=N_0}^{s} u_n\right| < \frac{3}{2}, \text{ for } s \geq N_0. \tag{14}$$

(Note: In the above inequalities we do not need the absolute value marks, but they are convenient later.)

Put $m = \frac{1}{2}|\prod_{n=p}^{N_0-1} u_n| > 0$, and $M = \frac{3}{2}|\prod_{n=p}^{N_0-1} u_n|$. Then

$$0 < m = \frac{1}{2}\left|\prod_{n=p}^{N_0-1} u_n\right| < \left|\prod_{n=N_0}^{s} u_n\right| \cdot \left|\prod_{n=p}^{N_0-1} u_n\right| = |\pi(p,s)|,$$

i.e., $0 < m < |\pi(p,s)|$ for all $s \geq N_0$. This says that $\prod_{n=p}^{\infty} u_n$ cannot converge to 0. Similarly, for all $s \geq N_0$ we have

$$|\pi(p,s)| = |\prod_{n=p}^{N_0-1} u_n| \cdot |\prod_{n=N_0}^{s} u_n| < |\prod_{n=p}^{N_0-1} u_n| \cdot \frac{3}{2} = M.$$

So we now have

$$0 < m < |\pi(p,s)| < M \text{ for all } s \geq N_0.$$

To see that the infinite product converges, let $\epsilon > 0$ be given and use the hypothesis to choose N, $N > N_0$, such that $|1 - \prod_{n=r}^{s} u_n| < \frac{\epsilon}{M}$ whenever $s \geq r \geq N$. Multiply this inequality by $|\pi(p, r-1)|$ to obtain $|\pi(p, r-1) - \pi(p,s)| = |\pi(p, r-1)| \cdot |1 - \prod_{n=r}^{s} u_n| < M \cdot \frac{\epsilon}{M} = \epsilon$. Hence the sequence $\{\pi(p,s)\}_{s=p}^{\infty}$ is a Cauchy sequence and must converge. ■

Corollary 2.2 *If $\prod_{n=1}^{\infty} u_n$ converges, then $u_n \to 1$ as $n \to \infty$.*

Proof: Take $r = s$ in Theorem 2.1. ■

Note: The converse of Corollary 2.2 is false.

Note: The convergence of an infinite product cannot be affected by changing a finite number of nonzero factors to other nonzero factors.

We often write the factors of a product in the form $u_n = 1 + a_n$. So by Corollary 2.2, $a_n \to 0$ if $\prod_{n=1}^{\infty}(1 + a_n)$ converges. When this is done, we call *a_n the nth term of the product.*

Theorem 2.3 *Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbf{C}$. Then $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if for some $p \in \mathbf{N}$ the series $\sum_{n=p}^{\infty} log(1 + a_n)$ converges. (Here "log" denotes the principal branch of the complex logarithm, and it is implicit that $1 + a_n \neq 0$ for $n \geq p$.) Moreover, if $s = \sum_{n=p}^{\infty} log(1 + a_n) \in \mathbf{C}$, then*

$$\prod_{n=1}^{\infty}(1 + a_n) = e^s \cdot \prod_{n=1}^{p-1}(1 + a_n).$$

Proof: Suppose the series $\sum_{n=p}^{\infty} log(1+a_n)$ converges to s. Since exp is continuous at s, it follows that $0 \neq e^s = lim_{q\to\infty} exp\left(\sum_{n=p}^{q} log(1+a_n)\right) = \lim_{q\to\infty} \prod_{n=p}^{q} exp(log(1+a_n)) = lim_{q\to\infty}\{\prod_{n=p}^{q}(1+a_n)\}$. So the infinite product converges to the asserted value.

Conversely, suppose the product converges and let $\epsilon > 0$ be given, $\epsilon < 1$. By the Cauchy Criterion there is a $p \in \mathbf{N}$ such that

$$\left|1 - \prod_{n=r}^{s}(1+a_n)\right| < \frac{\epsilon}{2} < \frac{1}{2} \text{ whenever } s \geq r \geq p. \tag{15}$$

Using the Taylor series $-log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$, we have

$$|log(1-z)| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n} \leq \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1-|z|} \leq 2|z| \text{ when } |z| \leq \frac{1}{2}. \tag{16}$$

Combine Eqs. 15 and 16 to obtain for $s \geq r \geq p$,

$$\begin{aligned} \left|log\left[\prod_{n=r}^{s}(1+a_n)\right]\right| &= \left|log\left(1 - \left(1 - \prod_{n=r}^{s}(1+a_n)\right)\right)\right| \\ &\leq 2\left|1 - \prod_{n=r}^{s}(1+a_n)\right| < \epsilon. \end{aligned} \tag{17}$$

In particular,

$$|log(1+a_n)| < \epsilon \text{ for all } n \geq p. \tag{18}$$

Recall that for real numbers z and w, $log(zw) = log(z) + log(w)$, but for complex z and w the best we can say is that there is some integer k for which $log(z) + log(w) = log(zw) + 2k\pi i$.

We want to prove that

$$\left|\sum_{n=r}^{s} log(1+a_n)\right| < \epsilon \text{ for } s \geq r \geq p. \tag{19}$$

This will show that $\sum_{n=1}^{\infty} log(1+a_n)$ is Cauchy and hence converges.

For fixed r, if $s = r$, then Eq. 19 follows from Eq. 18. Now suppose that Eq. 19 holds for some fixed s, $s \geq r$. We can choose an integer k such that

$$\sum_{n=r}^{s+1} log(1+a_n) = log\left[\prod_{n=r}^{s+1}(1+a_n)\right] + 2k\pi i. \tag{20}$$

In view of Eq. 17 we would have Eq. 19 for $s+1$ if we knew $k=0$. But it follows from Eqs. 20, 17, 18 and 19 for s that

$$\begin{aligned}
2\pi|k| &= |2k\pi i| \\
&= \left|\sum_{n=r}^{s} log(1+a_n) + log(1+a_{s+1}) - log\prod_{n=r}^{s+1}(1+a_n)\right| \\
&\le \left|log\prod_{n=r}^{s+1}(1+a_n)\right| + \left|\sum_{n=r}^{s} log(1+a_n)\right| + |log(1+a_{s+1})| \\
&< \epsilon+\epsilon+\epsilon \\
&< 3 < 2\pi,
\end{aligned}$$

implying that $k=0$. So $\sum_{n=p}^{\infty} log(1+a_n)$ converges. By the first part of the proof it converges to the desired value. ■

Corollary 2.4 *The series* $\sum_{n=1}^{\infty} b_n$ *of real numbers converges if and only if the infinite product* $\prod_{n=1}^{\infty} e^{b_n}$ *converges, in which case*

$$e^{\left[\sum_{n=1}^{\infty} b_n\right]} = \prod_{n=1}^{\infty} e^{b_n}.$$

Proof: In Theorem 2.3, $1+a_n \neq 0$ for all $n \geq 1$. Put $b_n = log(1+a_n)$. ■

For products having real terms of constant sign, the comparison with series is somewhat simpler.

Theorem 2.5 *Let* $\{a_n\}_{n=1}^{\infty} \subseteq [0,\infty]$. *Then* $\prod_{n=1}^{\infty}(1+a_n)$, $\prod_{n=1}^{\infty}(1-a_n)$, $\sum_{n=1}^{\infty} a_n$, *either all converge or all diverge.*

Proof: We may suppose that $a_n \to 0$ as $n \to \infty$, for otherwise all three diverge. Summands 0 or factors 1 cannot affect convergence and neither can a finite number of changes of summands or factors. Hence we also assume $0 < a_n < 1$ for all n. Since (using L'Hôpital on each of the three limits)

$$lim_{x\downarrow 0}\frac{log(1+x)}{x} = lim_{x\downarrow 0}\frac{-log(1-x)}{x} = lim_{x\downarrow 0}\frac{-log(1-x)}{log(1+x)} = 1,$$

it follows from the usual limit comparison test for series that the three series $\sum_{n=1}^{\infty} log(1+a_n)$, $-\sum_{n=1}^{\infty} log(1-a_n)$, $\sum_{n=1}^{\infty} a_n$ either all converge or all diverge. So the proof is completed by invoking theorem 2.3. ■

(Recall one version of the limit comparison test: If $a_n, b_n > 0$ and $lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.) The hypothesis that $a_n \geq 0$ for all n in Theorem 2.5 is crucial. Consider the following examples.

Example 6. Put $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$. Then $\sum a_n$ converges by the alternating series test, but we show that $\prod(1+a_n)$ diverges. $(1+a_{2k-1})(1+a_{2k}) = \left(1+\frac{1}{\sqrt{2k-1}}\right)\left(1-\frac{1}{\sqrt{2k}}\right) = 1-b_k$, where $0 < b_k < 1$ and $kb_k \to \frac{1}{2}$ as $k \to \infty$. By the limit comparison test $\sum b_k$ diverges (because $\sum \frac{1}{k}$ diverges). So $\prod(1-b_k)$ diverges by Theorem 2.5. But clearly $\prod_{k=1}^{n}(1-b_k)$ is monotonically decreasing as $n \to \infty$, so $\prod_{k=1}^{\infty}(1-b_k) \to 0$ as $n \to \infty$. So $\prod_{n=1}^{2m}(1+a_n) = \prod_{n=1}^{m}(1-b_n) \to 0$. Also, $\prod_{n=1}^{2m+1}(1+a_n) = \left(\prod_{n=1}^{2m}(1+a_n)\right)(1+a_{2m+1}) = \left[\prod_{n=1}^{m}(1-b_n)\right](1+\frac{1}{\sqrt{2m+1}}) \to 0$. So $\prod(1+a_n)$ diverges.

Example 7. Consider $\prod(1+a_n) = (1-\frac{1}{\sqrt{2}})(1+\frac{1}{\sqrt{2}}+\frac{1}{2})(1-\frac{1}{\sqrt{3}})(1+\frac{1}{\sqrt{3}}+\frac{1}{3})\cdots$ in which $a_{2n-1} = \frac{-1}{\sqrt{n+1}}$, $a_{2n} = \frac{1}{\sqrt{n+1}} + \frac{1}{n+1}$. Clearly $a_n \to 0$, $a_{2n-1}+a_{2n} = \frac{1}{n+1}$, $(1+a_{2n-1})(1+a_{2n}) = 1-\frac{1}{(n+1)^{3/2}}$. So $\sum a_n = \infty$, but $\prod(1+a_n)$ converges (use Theorem 2.5 again).

In spite of these examples we have the following fact.

Theorem 2.6 *Suppose that* $\{a_n\}_{n=1}^{\infty} \subset \mathbf{C}$ *satisfies* $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. *Then* $\sum_{n=1}^{\infty} a_n$ *and* $\prod_{n=1}^{\infty}(1+a_n)$ *either both converge or both diverge.*

Proof: Choose $n \in \mathbf{N}$ so that $|a_n| < 1$ for $n \geq N$. For $|z| < 1$,

$$log(1+z) = z + z^2[-\frac{1}{2} + \frac{z}{3} - \frac{z^2}{4} + - \cdots]. \qquad (21)$$

The power series in brackets represents a function continuous at $z = 0$, so it approaches $-\frac{1}{2}$ as z approaches 0. Put $z = a_n$ to obtain

$$b_n = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} a_n^k}{k+2} \to -\frac{1}{2} \text{ as } n \to \infty. \qquad (22)$$

Hence the sequence $\{b_n\}$ is bounded. So apply the hypothesis and Theorem 1.6 to obtain

$$\sum_{n=N}^{\infty} a_n^2 b_n \text{ converges (in fact, absolutely).} \qquad (23)$$

From Eqs. 21 and 22 we have

$$log(1+a_n) = a_n + a_n^2 b_n \text{ for } n \geq N. \tag{24}$$

It follows from Eqs. 23 and 24 that $\sum_{n=N}^{\infty} log(1+a_n)$ and $\sum_{n=N}^{\infty} a_n$ both converge or both diverge. This and Theorem 2.3 complete the proof. ■

Just as for series, there is a notion of absolute convergence for products.

Theorem 2.7 *If $\{a_n\}_{n=1}^{\infty} \subseteq \mathbf{C}$, then the following are equivalent:*

(i) $\sum_{n=1}^{\infty} |a_n| < \infty$*;*

(ii) $\prod_{n=1}^{\infty}(1+|a_n|) < \infty$.

(iii) $\sum_{n=p}^{\infty} |log(1+a_n)| < \infty$ *for some* $p \in \mathbf{N}$ $(a_n \neq -1$ *for* $n \geq p)$.

Defn. If these three hold, we say $\prod_{n=1}^{\infty}(1+a_n)$ is *absolutely convergent.*

Proof: The equivalence of (i) and (ii) follows from Theorem 2.5. If any of the three hold, then $a_n \to 0$. ($log(1+a_n) \to 0$ implies $1+a_n = exp(log(1+a_n)) \to e^0 = 1$.) So we may suppose p is large enough to force $|a_n| < \frac{1}{2}$ for $n \geq p$. We may also suppose $a_n \neq 0$ for all n. So

$$\begin{aligned} |a_n^{-1} log(1+a_n) - 1| &= |a_n^{-1}[a_n + a_n^2(-\frac{1}{2} + \frac{a_n}{3} - \frac{a_n^2}{4} + \cdots)] - 1| \\ &= |-\frac{a_n}{2} + \frac{a_n^2}{3} - \frac{a_3^3}{4} + \cdots| \\ &< \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \\ &= \frac{1}{2}. \end{aligned} \tag{25}$$

This implies $\frac{1}{2} < |[log(1+a_n)]/a_n| < \frac{3}{2}$. To finish the proof that (i) and (iii) are equivalent, just apply the limit comparison test, Theorem 1.11. ■

Corollary 2.8 *If* $\prod(1+a_n)$ *is absolutely convergent, then it is convergent.*

Proof: Suppose $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent. So by Theorem 2.7, $\sum_{n=1}^{\infty} |a_n| < \infty$. By the basic comparison test, $\sum |a_n^2| < \infty$. Then by Theorem 2.6, since $\sum |a_n| < \infty$ implies $\sum a_n$ converges, also $\prod(1+a_n)$ converges. ■

Theorem 2.9 *(The limit of a sum (resp., product)is the sum (resp. product) of the limits.)*

For $m, k \in \mathbf{N}$, *let* $a_m(k) \in \mathbf{C}$. *For each* $n \in \mathbf{N}$ *suppose that* $lim_{k\to\infty} a_n(k) = A_n \in \mathbf{C}$, *and suppose that* $|a_n(k)| \leq M_n < \infty$ *for all* n *and* k, *where* $\sum_{n=1}^{\infty} M_n < \infty$. *Then*

(i) $\lim_{k\to\infty}\left[\sum_{n=1}^{\infty} a_n(k)\right] = \sum_{n=1}^{\infty} A_n$, *and*

(ii) $lim_{k\to\infty} \prod_{n=1}^{\infty}(1 + a_n(k)) = \prod_{n=1}^{\infty}(1 + A_n)$,

where all sums and products appearing in (i) and (ii) converge in $\mathbf{C}$.

Proof: Let $\epsilon > 0$ be given.

From $\lim_{k\to\infty} a_n(k) = A_n$ and $|a_n(k)| \leq M_n$ we have that $|A_n| \leq M_n$. Since $\sum_{n=1}^{\infty} M_n < \infty$, we see that $\sum A_n$ converges absolutely. Hence there is an integer N_1 large enough so that for $m \geq N_1$ we have $|\sum_{n=m}^{\infty} A_n| < \epsilon/4$. Since $\sum M_n < \infty$, there is an integer N_2 such that for $m \geq N_2$ we have $\sum_{n=m}^{\infty} M_n < \epsilon/4$. It now follows that

$$\left|\sum_{n=m}^{\infty} a_n(k)\right| \leq \sum_{n=m}^{\infty} |a_n(k)| \leq \sum_{n=m}^{\infty} M_n < \epsilon/4.$$

Put $N = \max\{N_1, N_2\}$. For each $n \in \{1, 2, \ldots, N\}$ there is a k_n such that $|a_n(k) - A_n| < \frac{\epsilon}{2^{n+1}}$ for all $k \geq k_n$. Put $K = \max\{k_1, k_2, \ldots, k_N\}$. Then for all $k \geq K$ we have $|a_n(k) - A_n| < \frac{\epsilon}{2^{n+1}}$. Hence for all $k \geq K$ we have

$$\left|\sum_{n=1}^{\infty} a_n(k) - \sum_{n=1}^{\infty} A_n\right|$$

$$= \left|\sum_{n=1}^{\infty} a_n(k) - \sum_{n=1}^{N} a_n(k) + \sum_{n=1}^{N} a_n(k) - \sum_{n=1}^{N} A_n + \sum_{n=1}^{N} A_n - \sum_{n=1}^{\infty} A_n\right|$$

$$< \epsilon/4 + \sum_{n=1}^{N} \frac{\epsilon}{2^{n+1}} + \epsilon/4 < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.$$

This establishes part (i).

To prove (ii), first choose m such that $M_n < \frac{1}{2}$ for all $n > m$. Define $b_n(k) = log(1 + a_n(k))$, and $B_n = log(1 + A_n)$ for $n > m$ and all k. From the continuity of log on the disk $|1 - z| < 1$, it follows that $lim_{k\to\infty} b_n(k) =$

$lim_{k\to\infty} log(1+a_n(k)) = log(lim_{k\to\infty}(1+a_n(k)) = log(1+A_n) = B_n$ for $n > m$.

Recall that $log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{n}$. Then for $n > m$, $|b_n(k)| = \left|\sum_{j=1}^{\infty} \frac{(-1)^{j-1}(a_n(k))^j}{j}\right| \le \sum_{j=1}^{\infty} \frac{|a_n(k)|^j}{j} \le M_n \sum_{j=0}^{\infty} \frac{M_n^j}{j+1} < M_n \sum_{j=0}^{\infty} 2^{-j} = 2M_n$. Since $\sum 2M_n$ also converges, we can apply the first part of the theorem to obtain

$$lim_{k\to\infty} \sum_{n=m+1}^{\infty} b_n(k) = \sum_{n=m+1}^{\infty} B_n. \tag{26}$$

Then

$$\begin{aligned}
\prod_{n=m+1}^{\infty} (1+A_n) &= exp\left(\sum_{n=m+1}^{\infty} B_n\right) && \text{(Theorem 2.3)} \\
&= exp\left(\lim_{k\to\infty} \sum_{n=m+1}^{\infty} b_n(k)\right) && \text{(Eq. 26)} \\
&= lim_{k\to\infty}\left(exp \sum_{n=m+1}^{\infty} b_n(k)\right) && \text{(continuity of exp)} \\
&= lim_{k\to\infty}\left\{\prod_{n=m+1}^{\infty}(1+a_n(k))\right\}.
\end{aligned}$$

So finally

$$\begin{aligned}
&lim_{k\to\infty}\left\{\prod_{n=m+1}^{\infty}(1+a_n(k))\right\} \\
&= lim_{k\to\infty}\left[\prod_{n=1}^{m}(1+a_n(k))\left(\prod_{n=m+1}^{\infty}(1+a_n(k))\right]\right. \\
&= lim_{k\to\infty}\prod_{n=1}^{m}(1+a_n(k)) \cdot lim_{k\to\infty}\prod_{n=m+1}^{\infty}(1+a_n(k)) \\
&= \prod_{n=1}^{m}(1+A_n) \cdot \prod_{n=m+1}^{\infty}(1+A_n) \\
&= \prod_{n=1}^{\infty}(1+A_n).
\end{aligned}$$

■

3 Applications

Application 3.1 *For fixed z, $z \in \mathbf{C} \setminus \{0\}$, the infinite product*

$$\prod_{n=1}^{\infty} \frac{sin(z/n)}{z/n}$$

is absolutely convergent.

Proof: Recall $sin(w) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$. So

$$\frac{sin(z/n)}{z/n} = \sum_{k=0}^{\infty} \frac{(-1)^k (z/n)^{2k+1}}{(z/n)(2k+1)!}$$

$$= 1 + \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{n^{2(k-1)}(2k+1)!} = 1 + b_n/n^2,$$

where $b_n = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{n^{2(k-1)}(2k+1)!}$. For fixed $z \neq 0$ and any n,

$$|b_n| \leq \left| \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right| \cdot \frac{1}{|z|} \leq \left| \frac{sin(z) - z}{z} \right|.$$

So $\{|b_n|\}$ is a bounded sequence and $\sum \frac{1}{n^2}$ converges. By Theorem 1.6 $\sum \left|\frac{b_n}{n^2}\right|$ converges. Hence by Theorem 2.7 $\prod \left(1 + \left|\frac{b_n}{n^2}\right|\right)$ converges, which is equivalent to saying $\prod \left(1 + \frac{b_n}{n^2}\right) = \prod \frac{sin(z/n)}{z/n}$ converges absolutely. ■

Application 3.2 *For all $z \in \mathbf{C}$, $lim_{k\to\infty}(1 + (z/k))^k = e^z$.*

Proof: Fix $z \in \mathbf{C}$ and define $a_1(k) = 1 + z$ for all $k \in \mathbf{N}$ and

$$a_n(k) = \begin{cases} \binom{k}{n} z^n/k^n, & 1 < n \leq k; \\ 0, & 1 \leq k < n. \end{cases}$$

Then $A_n = \lim_{k\to\infty} a_n(k) = \lim_{k\to\infty} \frac{z^n}{n!} \cdot \frac{k(k-1)\cdots(k-n+1)}{k^n} = \frac{z^n}{n!}$ for $n > 1$. And $A_1 = lim_{k\to\infty} a_1(k) = 1 + z$. Also $|a_1(k)| \leq 1 + |z| = M_1 |a_n(k)| \leq \frac{|z|^n}{n!} = M_n$ for $n > 1$. And $\sum_{n=1}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$. Since $\sum_{n=1}^{\infty} a_n(k) = 1 + z + \sum_{n=2}^{k} \binom{k}{n} \left(\frac{z}{k}\right)^n = \left(1 + \frac{z}{k}\right)^k$, by part (i) of Theorem 2.9. We have $lim_{k\to\infty} \left(1 + \frac{z}{k}\right)^k = lim_{k\to\infty} \sum_{n=1}^{\infty} a_n(k) = \sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{z^n}{n!} = e^z$. ■

Application 3.3 *(Euler, 1748). For all $z \in \mathbf{C}$,*

$$sin(\pi z) = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Proof: For $m \in \mathbf{N}$, let

$$P_m(z) = \frac{1}{2i}\left[\left(1+\frac{\pi i z}{m}\right)^m - \left(1-\frac{\pi i z}{m}\right)^m\right].$$

From Application 3.2 we have

$$\begin{aligned} lim_{m\to\infty}P_m(z) &= \frac{1}{2i}\left[e^{\pi i z} - e^{-\pi i z}\right] \\ &= \frac{1}{2i}\left[(cos(\pi z) + isin(\pi z)) - (cos(-\pi z) + isin(-\pi z))\right] \\ &= \frac{1}{2i}\left[0 + 2i\ sin(\pi z)\right] = sin(\pi z). \end{aligned}$$

Hence

$$lim_{m\to\infty}P_m(z) = sin(\pi z) \text{ for all } z \in \mathbf{C}. \tag{27}$$

The zeros of $P_m(z)$ are those z for which $\left(\frac{m+\pi i z}{m-\pi i z}\right)^m = 1 = e^{(2\pi i)j}$, which holds if and only if $\frac{m+\pi i z}{m-\pi i z} = e^{\frac{2j\pi i}{m}}$ if and only if $(m+\pi i z) = (m-\pi i z)e^{\frac{2j\pi i}{m}}$ if and only if $z(\pi i + \pi i e^{\frac{2j\pi i}{m}}) = me^{\frac{2j\pi i}{m}} - m$ if and only if $z = \frac{m}{\pi i}\left(\frac{e^{\frac{2j\pi i}{m}}-1}{e^{\frac{2j\pi i}{m}}+1}\right) = \frac{m}{\pi}tan\frac{j\pi}{m}$. (It is an easy exercise to show that $\frac{e^{2i\theta}-1}{e^{2i\theta}+1} = i\ tan\theta$.) So $z = \frac{m}{\pi}tan\frac{j\pi}{m}$ are zeros of $P_m(z)$ for all integers j. If $m = 2k$ is even, then $P_m(z)$ has degree $m-1 = 2k-1$, and its distinct zeros are $0,\ \pm\left(\frac{2k}{\pi}\right)tan\left(\frac{j\pi}{2k}\right),\ 1 \le j < k$. (Recall that the tangent function is one-to-one on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.) The coefficient of z in $P_m(z)$ is easily seen to be π (and the constant term is zero), so

$$P_m(z) = P_{2k}(z) = \pi z \prod_{j=1}^{k-1}\left(1 - \frac{\pi^2 z^2}{4k^2 tan^2\left(\frac{j\pi}{2k}\right)}\right), \quad \text{for all } z \in \mathbf{C}. \tag{28}$$

Fix $z \in \mathbf{C}$ and define

$$a_j(k) = \begin{cases} \frac{-\pi^2 z^2}{4k^2 tan^2\left(\frac{j\pi}{2k}\right)}, & 1 \le j < k; \\ 0, & 1 \le k \le j. \end{cases}$$

From Eqs. 27 and 28

$$sin(\pi z) = lim_{m\to\infty}P_m(z)$$

$$
\begin{aligned}
&= lim_{k\to\infty} P_{2k}(z) \\
&= \pi z lim_{k\to\infty}\left(\prod_{j=1}^{\infty}(1+a_j(k))\right). \qquad (29)
\end{aligned}
$$

Since $x < tan\ x$ for $0 < x < \frac{\pi}{2}$ and $lim_{x\to 0}\frac{tanx}{x} = 1$, we have

$$
|a_j(k)| = \begin{cases} \frac{\pi^2|z|^2}{4k^2 tan^2\left(\frac{j\pi}{2k}\right)} \\ \text{or } 0 \end{cases} < \frac{\pi^2|z|^2}{4k^2\frac{j^2\pi^2}{4k^2}} = \frac{|z|^2}{j^2} = M_j.
$$

And $lim_{k\to\infty} a_j(k) = \frac{-z^2}{j^2} = A_j$. So by part (ii) of Theorem 2.9 ,

$$
\pi z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right) = \pi z\prod_{j=1}^{\infty}(1+A_j) =
$$

$$
= \pi z\ lim_{k\to\infty}\prod_{j=1}^{\infty}(1+a_j(k)) = sin(\pi z).
$$

■

We now have

$$
\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{sin(\pi z)}{\pi z} = 1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \frac{\pi^6 z^6}{7!} + \cdots. \qquad (30)
$$

Recall that $sinh\,\theta = \frac{e^{\theta}-e^{-\theta}}{2}$, from which it follows that $sin(iz) = i\ sinh(z)$. Replace z with iz in Eq. 30 to obtain:

Application 3.4

$$
\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right) = \frac{sin(\pi iz)}{\pi iz} =
$$

$$
= \frac{sinh(\pi z)}{\pi z} = 1 + \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} + \frac{\pi^6 z^6}{7!} + \cdots.
$$

Application 3.5 *Bernoulli Numbers and* $\zeta(2k)$

Define B_n, $n \geq 0$, by

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!}.
$$

The defining equation for B_n is equivalent to

$$1 = \left(\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}\right)\left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!}\right).$$

Recursively we can solve for the B_k using this equation. But first notice the following: Replace x by $-x$ in the (exponential generating) function for B_n:

$$\sum_{k=0}^{\infty} B_k \frac{(-x)^k}{k!} = \frac{-x}{e^{-x}-1} = \frac{xe^x}{e^x-1}.$$

So

$$\frac{x}{e^x-1} - \frac{xe^x}{e^x-1} = -x = \sum_{k=0}^{\infty} B_k \left[\frac{1-(-1)^k}{k!}\right] x^k.$$

This implies that

$$-x = B_0 \cdot 0 + B_1 \cdot \frac{2}{1}x + B_2 \cdot 0 \cdot x^2 + B_3 \cdot \frac{2}{3!}x^3 + B_4 \cdot 0 \cdot x^4 + \cdots$$

which implies that

$$B_1 = -\frac{1}{2} \text{ and } B_{2k+1} = 0 \text{ for } k \geq 1.$$

Then recursively from above we find $B_0 = 1$; $B_1 = -\frac{1}{2}$; $B_2 = \frac{1}{6}$; $B_4 = -\frac{1}{30}$; $B_6 = \frac{1}{42}, \ldots,$.

A famous result of Euler (and the main goal of these notes) is the following:

Theorem 3.6

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^k \pi^{2k} \cdot 2^{2k-1}}{(2k-1)!} \left(\frac{-B_{2k}}{2k}\right), \quad k = 1, 2, \ldots.$$

Proof: First take the logarithm of both sides of

$$sinh(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) \quad (\text{for } x > 0).$$

From the left side we get

$$\begin{aligned} log\ sinh(\pi x) &= log\left[\left(\frac{e^{\pi x}-e^{-\pi x}}{2}\right)\right] \\ &= log\left[\frac{e^{\pi x}}{2}(1-e^{-2\pi x})\right] \\ &= log(1-e^{-2\pi x})+\pi x-log\ 2. \end{aligned}$$

On the right side we get (for $0 < x < 1$)

$$log\ \pi + log\ x + \sum_{n=1}^{\infty} log\left(1+\frac{x^2}{n^2}\right) =$$

$$= log\ \pi + log\ x + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k}}{kn^{2k}}\right).$$

(Use Theorem 2.3 or Corollary 2.4.) Sinced this double series is absolutely convergent for $0 < x < 1$, we may interchange the order of summation and obtain the equality

$$log(1-e^{-2\pi x})+\pi x - log\ 2 =$$

$$= log\ \pi + log\ x + \sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}x^{2k}}{k}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right] =$$

$$= log\ \pi + log\ x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k}}{k}\zeta(2k).$$

On both sides take the derivative with respect to x. On the right side we may differentiate term by term, since the resulting series is uniformly convergent in $0 < x < 1-\epsilon$ for any $\epsilon > 0$. (See differentiation of power series in Calculus II.) We obtain:

$$\frac{2\pi e^{-2\pi x}}{1-e^{-2\pi x}}+\pi = \frac{1}{x}+2\sum_{k=1}^{\infty}(-1)^{k+1}x^{2k-1}\zeta(2k).$$

Multiply throughout by x and then substitute $\frac{x}{2}$ for x:

$$\frac{x\pi e^{-\pi x}}{1-e^{-\pi x}}+\frac{\pi x}{2} = 1+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{2^{2k-1}}\cdot\zeta(2k)$$

$$\to \frac{\pi x}{e^{\pi x}-1}+\frac{\pi x}{2}=1+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cdot\zeta(2k)}{2^{2k-1}}x^{2k}.$$

The left side gives $\frac{\pi x}{2}+\sum_{k=0}^{\infty}B_k\frac{(\pi x)^k}{k!}$. Compare coefficients of positive even powers of x:
$\frac{B_{2k}\pi^{2k}}{(2k)!}=\frac{(-1)^{k+1}\zeta(2k)}{2^{2k-1}}$, which implies

$$\zeta(2k)=\frac{(-1)^k\pi^{2k}2^{2k-1}}{(2k-1)!}\left(\frac{-B_{2k}}{2k}\right). \tag{31}$$

■

Now using the values of B_2, B_4 and B_6 computed earlier, we have:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\zeta(2\cdot 2)=\frac{\pi^4}{90}.$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6}=\zeta(2\cdot 3)=\frac{\pi^6}{945}.$$

Bernoulli originally introduced the B_n to give a closed form formula for

$$S_n(m)=1^n+2^n+3^n+\cdots+m^n.$$

On the one hand

$$\frac{x(e^{mx}-1)}{e^x-1}=\left(\frac{x}{e^x-1}(e^{mx}-1)\right)=\left(\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}\right)\left(\sum_{j=1}^{\infty}\frac{m^jx^j}{j!}\right)=$$

$$=\sum_{n=0}^{\infty}\left[\sum_{i=1}^{m}\frac{B_{n-i}}{(n-i)!}\cdot\frac{m^i}{i!}\right]x^n=\sum_{n=0}^{\infty}\left(\sum_{i=1}^{n}\binom{n}{i}B_{n-i}m^i\right)\frac{x^n}{n!}.$$

(The coefficient on $\frac{x^0}{0!}$ is 0.)

On the other hand:

$$\frac{x(e^{mx}-1)}{e^x-1}=x\left(\frac{e^{mx}-1}{e^x-1}\right)=x(e^{(m-1)x}+e^{(m-2)x}+\cdots+e^x+1)=$$

$$= x\sum_{j=0}^{m-1}\left(\sum_{r=0}^{\infty}\frac{j^r x^r}{r!}\right) = \sum_{r=0}^{\infty}\frac{x^{r+1}}{r!}\left(\sum_{j=0}^{m-1} j^r\right) =$$

$$= \sum_{r=0}^{\infty} S_r(m-1)\frac{x^{r+1}}{r!} = \sum_{n=1}^{\infty} S_{n-1}(m-1)\frac{nx^n}{n!}.$$

Equating the coefficients of $\frac{x^n}{n!}$ we get:

$$\sum_{i=1}^{n}\binom{n}{i} B_{n-i} m^i = S_{n-1}(m-1)n, \ n \geq 1,$$

or

$$\sum_{i=1}^{n+1}\binom{n+1}{i} B_{n+1-i}(m+1)^i = S_n(m)\cdot(n+1), \ n \geq 0.$$

So Bernoulli's formula is:

$$S_n(m) = 1^n + 2^n + \cdots + m^n = \sum_{i=1}^{n+1}\binom{n+1}{i} B_{n+1-i}\frac{(m+1)^i}{(n+1)}.$$