Trigonometric Identities and Inequalities: A Preview of Fourier Analysis

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Chapter 1

The Most Basic Identities and Inequalities

The trigonometric functions, especially the *sine* and *cosine* functions, play central roles in analysis, especially applied analysis. They contribute to the solutions of all kinds of problems involving finite sums, infinite sums, integrals, differential equations, and more. In the early part of these notes we develop some identities and inequalities that have special usefulness later on but might seem a bit unmotivated. Hence a reader might want to begin reading with Chapter 2 and return to the early sections as needed. In these notes it is assumed that the reader has met sequences and series of numbers and functions before, but even so we review a number of results that we need and which may not have been covered in a beginning calculus course.

1.1 The Basic Identities

For the definitions and basic properties of the trigonometric functions, see a good trig book, or even a calculus text. For example, *Calculus* by G. Strang, Wellesley – Cambridge Press, 1991, has a very good review of basic trigonometry. For our purposes we are content to take on faith the fact that sine is an odd function, cosine is an even function, and the usual addition formulas hold, viz.:

$$\begin{aligned}
\sin(-\theta) &= -\sin(\theta) && (1.1)\\
\cos(-\theta) &= \cos(\theta) && (1.2)\\
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B) && (1.3)\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B) && (1.4)
\end{aligned}$$

Many other identities arise from these, for example $\sin(\theta+\pi/2)=\cos(\theta)$. In Eq. 1.4 put $B=-A$ to obtain the most famous trig identity (the Pythagorean identity):

$$1 = \sin^2(A)+\cos^2(A) \tag{1.5}$$

Put $B=A$ in Eq. 1.3 to get

$$\sin(2A) = 2\sin(A)\cos(A) \tag{1.6}$$

Then $B=A$ in Eq. 1.4 gives

$$\cos(2A) = \cos^2(A)-\sin^2(A) \tag{1.7}$$

which we remember as either

$$\sin^2(A)=\frac{1-\cos(2A)}{2} \tag{1.8}$$

or

$$\cos^2(A) = \frac{1+\cos(2A)}{2} \tag{1.9}$$

Replace B with $-B$ in Eqs. 1.3 and 1.4 and use Eqs. 1.1 and 1.2 to obtain the identities

$$\cos[(k\pm\frac{1}{2})\theta] = \cos(k\theta)\cos(\frac{\theta}{2})\mp\sin(k\theta)sin(\frac{\theta}{2}) \tag{1.10}$$

Now use Eq. 1.10 to solve for

$$\cos[(k-\frac{1}{2})\theta] - \cos[(k+\frac{1}{2})\theta] \quad = \quad 2\sin(k\theta)\sin(\frac{\theta}{2}) \tag{1.11}$$

Note that if $\theta \neq 2k\pi$ for some $k \in \mathcal{N}$, then $\sin(\theta/2) \neq 0$, so we obtain.

$$\sin(k\theta) \quad = \quad \frac{\cos[(k-\frac{1}{2})\theta] - \cos[(k+\frac{1}{2})\theta]}{2\sin(\frac{\theta}{2})} \tag{1.12}$$

The following telescoping sum then becomes

$$S_n(\theta) = \sum_{k=1}^{n} \sin(k\theta) \quad = \quad \frac{\cos(\frac{\theta}{2}) - \cos[(n+\frac{1}{2})\theta]}{2\sin(\frac{\theta}{2})} \tag{1.13}$$

From Eq. 1.13 we see that

$$|S_n(\theta)| \quad \leq \quad \frac{1+1}{|2\sin(\frac{\theta}{2})|} = \frac{1}{|\sin(\frac{\theta}{2})|}. \tag{1.14}$$

This shows that

Corollary 1.1.1. *$S(\theta) = \sum_{k=1}^{\infty} \sin(k\theta)$ has bounded partial sums.*

We now give an analog for a sum of cosines using a sequence $\{D_n\}$ of functions known as the **Dirichlet kernel**.

1.2 The Dirichlet Kernel: A first look

For each positive integer n define

$$D_n(t) = \begin{cases} \frac{2n+1}{2\pi} & \text{if } t = 2\pi m,\ m \in Z; \\ \frac{\sin[(n+\frac{1}{2})t]}{2\pi\sin(t/2)} & \text{if } t \neq 2\pi m. \end{cases}$$

Lemma 1.2.1.

$$D_n(u) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{n}\cos(ku)\right).$$

Proof. If $u = 2\pi m$, the result is clear. So suppose $u \neq 2\pi m$. Then

$$\begin{aligned}
\sin(\frac{u}{2})\left(1 + 2\sum_{k=1}^{n}\cos(ku)\right) &= \sin(\frac{u}{2}) + \sum_{k=1}^{n} 2\sin(\frac{u}{2})\cos(ku) \\
&= \sin(\frac{u}{2}) + \sum_{k=1}^{n}\left(\sin[k+\frac{1}{2})u] - \sin[(k-\frac{1}{2})u]\right) \\
&= \sin[(n+\frac{1}{2})u] \\
&= 2\pi\sin(\frac{u}{2})\left(\frac{\sin[(n+\frac{u}{2})u]}{2\pi\sin(\frac{u}{2})}\right).
\end{aligned}$$

□

By the above lemma it is clear that $D_n(u)$ is 2π-periodic and continuous except possibly at multiples of 2π. Moreover, $D_n(u)$ is an even function. To see that $D_n(u)$ is continuous at $2\pi m$, it suffices to see that it is continuous at 0. So compute

$$\lim_{t\to 0} D_n(t) = \lim_{t\to 0}\frac{\sin[(n+\frac{1}{2})t]}{t}\frac{t}{2\pi\sin(\frac{t}{2})} = \frac{n+\frac{1}{2}}{2\pi/2} = \frac{2n+1}{2\pi}.$$

1.3 Some Basic Inequalities

Probably the best known inequality is the following one.

Lemma 1.3.1.

$$|\sin(x)| < |x| \quad \textit{for all } x \neq 0.$$

Proof. Define a function $f(x) = x - \sin(x)$. Note that $f(0) = 0$ and $f'(x) = 1 - \cos(x) \geq 0$ with equality only at multiples of 2π. So on $[0, \infty)$ the function $f(x)$ is increasing. So $f(x) > 0$ for $0 < x < \infty$. And for $x > 0$, $f(-x) = (-x) - \sin(-x) = -(x - \sin(x)) < 0$. It follows that for all nonzero real x we have $|\sin(x)| < |x|$. □

Lemma 1.3.2. *For all $x \in [-\pi, \pi]$ we have $|x| \leq \pi \cdot |\sin(x/2)|$.*

Proof. Define a function $f(x) = \pi\sin(x) - 2x$ and note that $f(0) = f(\pi/2) = 0$. Here $f'(x) = \pi\cos(x) - 2$ and $f''(x) = -\pi\sin(x)$. So $f'(x)$ is strictly decreasing on $[0, \pi/2]$. Since $f'(0) = \pi - 2 > 0$ and $f'(\pi/2) = -2 < 0$ there

is a unique $x_0 \in (0, \pi/2)$ with $f'(x_0) = 0$, $f(x)$ is increasing on $[0, x_0]$ and decreasing on $[x_0, \pi/2]$. It follows that $f(x) > 0$ for $x \in (0, \pi/2)$. Since $f(x)$ is an odd function (i.e., $f(-x) = -f(x)$, we have $|2x| \leq \pi|\sin(x)|$ for $x \in [-\pi/2, \pi/2]$, which is equivalent to $|x| \leq \pi|\sin(x/2)|$ for $x \in [-\pi, \pi]$. $\square$

As an immediate corollary we have:

Corollary 1.3.3. *If ϵ is chosen so that $0 < \epsilon/2 < \pi - \epsilon/2 < \pi$, then for each $x \in [\epsilon/2, \pi - \epsilon/2]$ it is true that*

$$\sin(x) \geq \frac{2x}{\pi} > 0.$$

In the proof of the next lemma we use Taylor's theorem with integral remainder.

Theorem 1.3.4. *Let f be a real-valued function defined on an interval I and having an $n+1$-st derivative that is continuous on I. Let c be a point in the interior of I and define the Taylor polynomial for f of degree n at c by*

$$P_n(x-c) = \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Then

$$f(x) = P_c(x) + R_n(x) \text{ for any } x \in I \text{ where } R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n dt.$$

Proof. The key to the proof is to hold x fixed, and to regard c as a variable. So consider the function

$$g(t) = P_t(x) = f(t) + f'(t)(x-t) + f''(t)\frac{(x-t)^2}{2!} + \cdots + f^{(n)}(t)\frac{(x-t)^n}{n!}.$$

Note that $g(x) = f(x)$ and $g(c) = P_c(x)$. Hence

$$R_n(x) = f(x) - P_c(x) = g(x) - g(c) = \int_c^x g'(t)dt.$$

At this point we need only to calculate g'. Hence:

$$\begin{aligned} g'(t) &= f'(t) + \{f''(t)(x-t) - f'(t)\} \\ &+ \left\{ f^{(3)}(t)\frac{(x-t)^2}{2!} - f''(t)\frac{(x-t)}{1!} \right\} \\ &+ \cdots\cdots \\ &+ \left\{ f^{(n+1)}(t)\frac{(x-t)^n}{n!} - f^{(n)}(t)\frac{(x-t)^{(n-1)}}{(n-1)!} \right\}. \end{aligned}$$

This is a telescoping sum that results in the simple result

$$g'(t) = f^{(n+1}(t)\frac{(x-t)^n}{n!}.$$

□

Lemma 1.3.5. *If* $|x| \leq \pi$, *then* $1 - \frac{1}{2}x^2 \leq cos(x) \leq 1 - \frac{1}{5}x^2$.

Proof. Since all functions involved are even, we may assume $0 \leq x \leq \pi$. By Taylor's theorem, $f(x) = \cos(x) = P_n(x) + \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n dt$, where P_n is the Taylor polynomial for cos of degree n at $c = 0$. For $n = 2$, we have $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2}\int_0^x \sin(t)\cdot(x-t)^2 dt$. Since $\frac{1}{2}\int_0^x \sin(t)\cdot(x-t)^2 dt \geq 0$, the first inequality holds. Consider the function $g(x) = 1 - \frac{1}{5}x^2 - \cos(x) = \frac{3}{10}x^2 - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots$, $0 \leq x \leq \pi$. $g(0) = 0$, $g(\pi) = 2 - \frac{\pi^2}{5} \sim .03 > 0$. (Alternatively, $2 - \frac{\pi^2}{5} > 0$ if and only if $10 > \pi^2$ if and only if $\sqrt{10} > \pi$. If we cheat and use a calculator we see that $\sqrt{10} > 3.16$, which is clearly greater than $\pi \sim 3.14159$.) So if $g(x) < 0$ for some $x \in (0, \pi)$, let x_0 be the point in $(0, \pi)$ at which $g(x)$ takes on its minimum value. Then $g'(x_0) = 0$, and $g''(x_0) > 0$ since $g''(x_0)$ is a local minimum. Here $g'(x) = -\frac{2}{5}x + \sin(x)$, so $\sin(x_0) = \frac{2}{5}x_0$. And $\sin^2(x_0) = \frac{4}{25}x_0^2$. Also, $g''(x_0) = -\frac{2}{5} + \cos(x_0) > 0$ implies $\cos(x_0) > \frac{2}{5}$. And this implies $1 = \sin^2(x_0) + \cos^2(x_0) > \frac{4}{25}x^2 + \frac{4}{25}$, which forces $\frac{21}{4} > x_0^2$. Since $\frac{36}{5} > \frac{21}{4}$, we have $\frac{36}{5} > x_0^2$, which is equivalent to $\frac{3}{10}x_0^2 - \frac{x_0^4}{4!} > 0$. So the two terms of the series for $g(x)$ add up to a positive number. It is easy to check that each succeeding pair of terms adds up to a positive number: $\frac{x^{2k}}{(2k)!} - \frac{x^{(2k+2)}}{(2k+2)!} > 0$. Hence $g(x_0) > 0$, contradicting the assumption that $g(x)$ takes on a negative value at some point of $(0, \pi)$. This means that $g(x) \geq 0$ for all $x \in (0, \pi)$, i.e., $\cos(x) \leq 1 - \frac{1}{5}x^2$ for all $x \in (0, \pi)$, completing the proof of the lemma. □

1.4 A Divergent Sum

One of the main examples that will reappear several times throughout these notes is the infinite series of functions.

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^P}$$

where p is any fixed positive real number. Clearly it is necessary only to study the series for $x \in [-\pi, \pi]$. In order to study this series we need the following inequality.

Suppose that ϵ and θ are fixed positive real numbers satisfying the following:

$$2\epsilon < \theta < \pi - 2\epsilon \tag{1.15}$$

Our first goal in this section is to prove the following theorem.

Theorem 1.4.1.

$$\underline{\lim}_{n\to\infty} \frac{|\sin(\theta)| + |\sin(2\theta)| + \cdots + |\sin(n\theta)|}{n} \geq \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n-1}{2} \sin(\epsilon)$$

$$= \frac{\sin(\epsilon)}{2} > 0.$$

In order to prove this theorem we first establish the following lemma.

Lemma 1.4.2. *If* $|\sin(m\theta)| \leq \sin(\epsilon)$ *for some integer* m*, then*

$$|\sin[(m+1)\theta] > \sin(\epsilon).$$

Proof. For $u \in \mathcal{R}$, let $\overline{u}$ be the value of u modulo 2π, i.e., for a unique integer k, $u = 2k\pi + \overline{u}$ where $0 \leq \overline{u} < 2\pi$. First note that $|\sin(u)| \leq \sin(\epsilon)$ if and only if

(i) $0 \leq \overline{u} \leq \epsilon$, or

(ii) $\pi - \epsilon \leq \overline{u} \leq \pi + \epsilon$, or

(iii) $2\pi - \epsilon \leq \overline{u} < 2\pi$.

It follows that

$|\sin(u)| > \sin(\epsilon)$ precisely when $\epsilon < \overline{u} < \pi - \epsilon$ or $\pi + \epsilon < \overline{u} < 2\pi - \epsilon$.

Now suppose that $|\sin(u)| \leq \sin(\epsilon)$ and show that $|\sin(u+\theta)| > \sin(\epsilon)$.
Case (i). Add $0 \leq \overline{u} \leq \epsilon$ to Eq. 1.15 to obtain

$$\begin{aligned} 2\epsilon < \overline{u} + \theta < \pi - \epsilon \quad &\Longrightarrow \quad 2\epsilon < \overline{u+\theta} < \pi - \epsilon \\ &\Longrightarrow \quad |\sin(u+\theta)| > \sin(\epsilon). \end{aligned}$$

Case (ii). Add $\pi - \epsilon \leq \overline{u} \leq \pi + \epsilon$ to Eq. 1.15 to obtain

$$\begin{aligned} \pi + \epsilon < \overline{u} + \theta < 2\pi - \epsilon \quad &\Longrightarrow \quad \pi + \epsilon < \overline{u + \theta} < 2\pi - \epsilon \\ &\Longrightarrow \quad |\sin(u + \theta)| > \sin(\epsilon). \end{aligned}$$

Case (iii). Add $2\pi - \epsilon \leq \overline{u} < 2\pi$ to Eq. 1.15 to obtain

$$\begin{aligned} 2\pi + \epsilon < \overline{u} + \theta < 3\pi - 2\epsilon \quad &\Longrightarrow \quad \epsilon < \overline{u + \theta} < \pi - 2\epsilon \\ &\Longrightarrow \quad |\sin(u + \theta)| > \sin(\epsilon). \end{aligned}$$

If we put $u = m\theta$ we see that the Lemma is proved. □

The preceding Lemma shows that at least $\frac{n-1}{2}$ of the summands in

$$|\sin(\theta)| + |\sin(2\theta)| + \cdots + |\sin(n\theta)|$$

are greater than $\sin(\epsilon)$. It follows that

$$\begin{aligned} \underline{\lim}_{n\to\infty} \frac{|\sin(\theta)| + |\sin(2\theta)| + \cdots + |\sin(n\theta)|}{n} &\geq \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n-1}{2} \sin(\epsilon) \\ &= \frac{\sin(\epsilon)}{2} > 0. \end{aligned}$$

This completes a proof of Theorem 1.4.1.

The next lemma gives a test for divergence.

Lemma 1.4.3. *Let $\{a_n\}$ be a sequence with $0 \leq a_{n+1} < a_n$ for $n \geq k$ and let $\{w_n\}$ be a sequence of positive numbers such that*

$$\sum w_n(a_n - a_{n+1}) = \infty.$$

Suppose that $\{b_n\}_{n=k}^{\infty}$ is a sequence of real numbers such that

$$\underline{\lim}_{n\to\infty} \frac{b_k + b_{k+1} + \cdots + b_n}{w_n} = u > 0.$$

Then $\sum_{n=k}^{\infty} a_n b_n = \infty$.

Proof. Put $S_n = \sum_{j=k}^{n} a_j b_j$, $n \geq k$. So $\{S_n\}_{n=k}^{\infty}$ is the sequence of partial sums of the series $\sum_{n=k}^{\infty} a_n b_n$.

Put $B_n = b_k + b_{k+1} + \cdots + b_n$. Then $b_k = B_k$, $b_j = B_j - B_{j-1}$ for $j \geq k+1$.

And

$$\begin{aligned} S_n &= a_k B_k + a_{k+1}(B_{k+1} - B_k) + \cdots + a_n(B_n - B_{n-1}) \\ &= (a_k - a_{k+1})B_k + \cdots + (a_{n-1} - a_n)B_{n-1} + a_n B_n \\ &= T_{n-1} + a_n B_n, \quad \text{where } T_{n-1} = \sum_{j=k}^{n-1}(a_j - a_{j+1})B_j. \end{aligned}$$

By hypothesis there is an $N \in \mathcal{N}$ for which $\frac{b_k + \cdots + b_n}{w_n} > \frac{u}{2}$ for $n \geq N$, i.e., $B_n > w_n \cdot \frac{u}{2}$ for $n \geq N$. So for such n we have

$$\sum_{n=N}^{\infty} B_n(a_n - a_{n+1}) > \sum_{n=N}^{\infty} w_n \cdot \frac{u}{2}(a_n - a_{n+1}).$$

Since $\sum w_n(a_n - a_{n-1}) = \infty$, also $\sum_{n=N}^{\infty} B_n(a_n - a_{n+1}) = \infty$. Then $S_n = T_{n-1} + a_n B_n > T_{n-1}$, where the partial sum T_{n-1} of $\sum(a_j - a_{j+1})B_j$ must approach ∞. Hence $\{S_n\} \to \infty$, implying $\sum a_j b_j = \infty$. □

The divergent series we want to consider first is the following:

$$\sum_{n=1}^{\infty} \frac{|\sin(n\theta)|}{n}. \tag{1.16}$$

Put $w_n = n$ for $n \geq 1$; $a_n = \frac{1}{n}$; $b_n = |\sin(n\theta)|$. Then $0 < a_{n+1} < a_n$. Also

$$\begin{aligned} \sum w_n(a_n - a_{n+1}) &= n\left(\frac{1}{n} - \frac{1}{n+1}\right) = \\ \sum n\left(\frac{1}{n(n+1)}\right) &= \sum \frac{1}{n+1} = \infty. \end{aligned}$$

Reduce θ modulo $2\pi$to $\overline{\theta}$ where $\theta = 2k\pi + \overline{\theta}$ and $0 < \overline{\theta} < 2\pi$. Clearly $\sin(\theta) = \sin(\overline{\theta})$. And $\sin(n\overline{\theta}) = \sin(n\theta)$. If $\pi < \overline{\theta} < 2\pi$, replace $\overline{\theta}$ with $2\pi - \overline{\theta}$. Then $|\sin(n\theta)| = |-\sin(2n\pi - n\theta)| = |\sin(n(2\pi - \theta))|$, so in the sum $\sum_{n=k}^{\infty} |\sin(n\theta)|$ we may assume that $0 < \theta < \pi$. Now choose an ϵ such that $2\epsilon < \theta < \pi - 2\epsilon$, so we can apply Theorem 1.4.1 to see that

$$\underline{\lim}_{n\to\infty} \sum_{j=1}^{n} \frac{|\sin(j\theta)|}{n} \geq \frac{\sin(\epsilon)}{2} > 0.$$

It now follows by Lemma 1.4.3 that $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{|\sin(n\theta)|}{n}$ diverges. Since for $p \leq 1$ we have $\frac{1}{n^p} \geq \frac{1}{n}$, it follows that

Corollary 1.4.4. *If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{|\sin(n\theta)|}{n^p}$ diverges.*

1.5 Integrating even and odd functions

Let a and b be real numbers with $0 \leq b < a$. Suppose f is continuous on the intervals $[-a, -b]$ and $[b, a]$. If $f(-t) = f(t)$ for all $t \in [b, a]$ we say f is an *even* function on $I = [-a, -b] \cup [b, a]$. If $f(-t) = -f(t)$ for all $t \in [b, a]$, we say f is an *odd* function on I. The product of two even (resp., odd) functions is even, and the product of an odd function and an even function is an odd function. Suppose that f is a function defined and continuous on the interval I just given.

Lemma 1.5.1. *Using the change of variables $u = -t$ it is easy to show that*

(i) f *even* $\implies \int_{-a}^{-b} f(t)dt = \int_b^a f(t)dt.$

(ii) f *odd* $\implies \int_{-a}^{-b} f(t)dt = -\int_b^a f(t)dt.$

(iii) f *even* $\implies \int_{-\pi}^{\pi} f(t)\sin(nt)dt = 0,$ *for* $n \geq 1.$

(iv) f *odd* $\implies \int_{-\pi}^{\pi} f(t)\cos(nt)dt = 0,$ *for* $n \geq 0.$

(v) $\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0$ *for any nonnegative integers* $m, n.$

(vi) $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$ *if* $m \neq n.$

(vii) $\int_{-\pi}^{\pi} \sin(nt)\sin(nt)dt = \pi.$

(viii) $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = 0$ *if* $m \neq n.$

(ix) $\int_{-\pi}^{\pi} \cos(nt)\cos(nt)dt = \pi.$

(x) $\int_{-\pi}^{\pi} \cos(nt)dt = 0$ *if* $n \geq 1$.

(xi) $\int_{-\pi}^{\pi} \sin(nt)dt = 0$ *if* $n \geq 1$.

Proof. Once the general trick is learned, the proofs are all routine. We give one basic example and leave the others to the reader.

Start with the following identities that can be derived from Eqs. 1.3 and 1.4:

$$2\sin A \sin B = \cos(A-B) - \cos(A+B); \tag{1.17}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B); \tag{1.18}$$

$$2\cos A \cos B = \cos(A-B) + \cos(A+B). \tag{1.19}$$

Then for $n \neq m$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos((n-m)x) - \cos((n+m)x)]dx \\
&= \frac{1}{2}\left[\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m}\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

And for $n = m$,

$$\int_{-\pi}^{\pi} \sin(nx)\cdot\sin(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} [1-\cos(2nx)]\,dx = \pi.$$

Etc.)

□

1.6 Review Uniform Convergence

Recall that if f is a function which is defined on a set E, then $\|f\|_E$ denotes the supremum of the set of values $|f(p)|$ for $p \in E$. When f is continuous on E and E is closed and bounded (i.e., compact if $E \subseteq \mathcal{R}^n$), then this is the *maximum* value of $|f(p)|$ on E. If f and g are both defined on E, then $\|f-g\|_E$ is a measure of the distance between f and g over the set E. If $\|f-g\|_E < \epsilon$, then $|f(p)-g(p)| < \epsilon$ for all $p \in E$, so that f approximates g within ϵ uniformly on E.

Definition 1.6.1. *A sequence $\{f_n\}$ is uniformly convergent to a function F on a set E if and only if*

$$\lim_{n\to\infty} \|F - f_n\|_E = 0.$$

Recall the Weierstrass M-test:

Theorem 1.6.2. *The series $\sum f_n$ converges uniformly on a set S if*

$$\|f_n\|_S \leq M_n,\ n \geq k,\ \textit{where} \sum M_n < \infty.$$

The next two theorems really explain our interest in uniform convergence.

Theorem 1.6.3. *If $\{f_n\}$ converges to F uniformly on E and each function f_n is continuous on E, then F is continuous on E. Stated for series this becomes: If each of the functions f_n is continuous on a set E and $\sum f_n$ converges to F uniformly on E, then F is continuous on E.*

Theorem 1.6.4. *Let the functions f_n each be continuous on the closed bounded interval $[a, b]$. Then if $\{f_n\}$ converges uniformly to F on $[a, b]$,*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b F.$$

Stated for series this becomes: If each of the functions f_n is continuous on $[a, b]$ and $\sum f_n$ converges uniformly to F on $[a, b]$, then

$$\int_a^b \sum f_n = \sum \int_a^b f_n.$$

For $p > 1$ put $M_n = \frac{1}{n^p}$. Since in this case $\sum M_n$ converges, we have the following corollary:

Corollary 1.6.5. *If $p > 1$, the series $\sum \frac{Sin(nx)}{n^p}$ converges uniformly on $\mathcal{R}$. In fact, for each fixed $x \in \mathcal{R}$ the series converges absolutely.*

1.7 The Series $\sum \frac{\sin(n\theta)}{n^p}$

We have seen that if θ is arbitrary and $p > 1$ the series $\sum \frac{\sin(n\theta)}{n^p}$ converges absolutely. In this section we suppose that $0 < p \leq 1$.

In our next proof we need to use Abel's theorem:

Theorem 1.7.1. *(Abel's Test) The series $\sum a_n b_n$ converges if $a_{n+1} \leq a_n$ for $n \geq k$, $\lim_{n\to\infty} a_n = 0$, and*

$$|b_k + b_{k+1} + \cdots + b_n| \leq M, \ n \geq k,$$

for some constant M.

Lemma 1.7.2. *For $0 < p \leq 1$, $\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^p}$ converges pointwise on $\mathcal{R}$.*

Proof. Put $F_n(\theta) = \frac{\sin(n\theta)}{n^p}$. First note that if $\theta = 2k\pi$ for some integer k, then $F_n(\theta) = 0$ and the series converges pointwise trivially. So suppose that $\theta \neq 2k\pi$ for any integer k. Put $a_n = \frac{1}{n^p}$, so $0 < a_{n+1} < a_n$ and $\lim_{n\to\infty} a_n = 0$. If we put $b_n = \sin(n\theta)$ and $B_n = b_2 + b_3 + \cdots b_n$ for $n \geq 2$, then in Eq. 1.14 we showed that $|B_n|$ is bounded by $\left|\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right|$.

This says that (for fixed $\theta \neq 2k\pi$) the sequence $\{F_n(\theta)\}$ satisfies the hypothesis of Abel's test, which implies that $\sum \frac{\sin(n\theta)}{n^p}$ converges. □

If we put together Corollaries 1.4.4 and 1.6.5 and Lemma 1.7.2 we obtain the following.

Theorem 1.7.3. *Let θ be any real number not an integral multiple of 2π, and let $p > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^p} \quad \begin{cases} \textit{converges absolutely if } p > 1; \\ \textit{converges conditionally if } 0 < p \leq 1. \end{cases}$$

Since for $p > 1$ the series $\sum \frac{\sin(nx)}{n^p}$ converges uniformly on $\mathcal{R}$, it converges uniformly on each closed bounded interval $[a, b]$, implying that

$$\lim_{n\to\infty} \int_a^b \frac{\sin(nx)}{n^p} dx = \int_a^b \lim_{n\to\infty} \frac{\sin(nx)}{n^p} = \int_a^b \overline{0} dx = 0,$$

and

$$\sum_{n=1}^{\infty} \int_a^b \frac{\sin(nx)}{n^p} dx = \int_a^b \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^p} dx.$$

1.8 Uniform Convergence on a Proper Subinterval

Lemma 1.8.1. *Pick $\epsilon > 0$ so that $0 < \epsilon < 2\pi - \epsilon < 2\pi$, i.e., $\epsilon < \pi$. Then for $\theta \in [\epsilon, 2\pi - \epsilon]$ we have*

$$|S_n(\theta)| \leq \frac{\pi}{\epsilon}. \tag{1.20}$$

Proof. From Eq. 1.14 we know that $|S_n(\theta)| \leq \frac{1}{|\sin(\frac{\theta}{2})|}$. Then from

$$0 < \epsilon/2 \leq \theta/2 \leq \pi - \epsilon/2 < 2\pi$$

we have $\sin(\theta/2) \geq \sin(\epsilon/2) > 0$, hence

$$\frac{1}{|\sin(\theta/2)|} \leq \frac{1}{\sin(\epsilon/2)}. \tag{1.21}$$

By Cor. 1.3.3 we know that for $x \in [\epsilon/2, \pi - \epsilon/2]$ it is true that

$$\sin(x) \geq \frac{2}{\pi} x > 0. \tag{1.22}$$

Hence from Eqs. 1.14, 1.21 and 1.22 we find that Eq. 1.20 holds. □

Consider the (Fourier) sine series

$$f(\theta) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\theta). \tag{1.23}$$

We have already established (see Theorem 1.7.3) the fact that this series converges to a function $f(\theta)$. Since the convergence was not absolute, it is not uniform. But if we restrict the function to the interval $[\epsilon, 2\pi - \epsilon]$ we will be able to show that the convergence is uniform.

Notice that

$$\begin{aligned} \frac{1}{k} &= \left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{n} \\ &= \sum_{j=k}^{n-1} \frac{1}{j(j+1)} + \frac{1}{n}. \end{aligned}$$

This gives:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k}\sin(k\theta) &= \sum_{k=1}^{n}\left(\sum_{j=k}^{n-1}\frac{1}{j(j+1)}+\frac{1}{n}\right)\sin(k\theta)\\ &= \sum_{j=1}^{n-1}\frac{1}{j(j+1)}S_j(\theta)+\frac{1}{n}S_n(\theta).\end{aligned}$$

On the interval $[\epsilon, 2\pi-\epsilon]$ we have the bounds

$$\sup_{\epsilon\le\theta\le 2\pi-\epsilon}\left|\frac{1}{j(j+1)}S_j(\theta)\right| \le \frac{\pi}{j(j+1)\epsilon}.$$

So we can apply the Weierstrass M-test. We have a telescoping sum

$$\sum_{j=1}^{\infty}\frac{\pi}{j(j+1)\epsilon} = \frac{\pi}{\epsilon}\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right) = \frac{\pi}{\epsilon} < \infty.$$

Hence the series $\sum_{j=1}^{\infty}\frac{1}{j(j+1)}S_j(\theta)$ converges uniformly on $[\epsilon, 2\pi-\epsilon]$. Since the uniform limit of continuous functions is continuous, the sum is a continuous function on $[\epsilon, 2\pi-\epsilon]$ for any $\epsilon > 0$. Therefore the limit exists and is a continuous function $f(\theta)$ on $(0, 2\pi)$.

In Davidson and Donsig [DD02] there is an interesting investigation of this function. Using complex functions they determine the function to which the series converges on the interval $(0, 2\pi)$. After we have proved the Jordan-Dirichlet Theorem we find **using only real numbers** the function to which the series converges on the interval $[-\pi, \pi]$. Hence at this point we merely direct the reader to pages 458 – 461 of [DD02].

Chapter 2

Inner Product Spaces

2.1 Real (or Complex) Vector Spaces

Let F be any subfield of the complex numbers $\mathcal{C}$. For each positive integer n, the set $F^n = \{(a_1, \ldots, a_n) : a_i \in F, 1 \leq i \leq n\}$ together with coordinatewise vector addition and the usual scalar product (i.e., $c(a_1, \cdots, a_n) = (ca_1, \cdots, ca_n)$) yields the standard vector space studied in beginning linear algebra courses. Another example nearly as familiar is the polynomial ring $F[x]$ in the variable x. Here are two more examples central to analysis. First, for each positive integer p,

$$l_p = \left\{ \{a_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^p \text{converges}, a_i \in \mathcal{R}, i = 1, 2, \cdots \right\},$$

with "coordinatewise" addition and the scalar product $c \cdot \{a_n\} = \{ca_n\}$ is a vector space. The important case for our needs here is l_2, which is called real (separable) Hilbert space (when endowed with the inner product introduced in the next section). Finally, if $a < b$, then $C[a, b] = \{f : [a, b] \to \mathcal{R} : f \text{ is continuous on } [a, b]\}$ with the usual addition and scalar product is a real vector space.

All the above examples with $F = \mathcal{R}$ will be important in these notes.

2.2 Real (or Complex) Euclidean Space

As above, let F be any subfield of $\mathcal{C}$ and let V be any vector space over F. For $x \in F$, $\overline{x}$ denotes the complex conjugate of x. So $x = \overline{x}$ if and only if

$x \in \mathcal{R}$.

A function $(\ ,\ ) : V \times V \to F$ is a (Euclidean) *inner product* (and $(V, (\ ,\ ))$ is a *Euclidean space*) provided the following hold:

1. $(u, v + w) = (u, v) + (u, w)$, for all $u, v, w \in V$.
2. $(u, av) = a(u, v)$, for all $a \in F$, $u, v \in V$.
3. $(u, v) = \overline{(v, u)}$ for all $u, v \in F$.
4. $(u, u) \geq 0$, with equality if and only if $u = \overline{0} \in V$.

Note: From now on we assume that $(V, (\ ,\ ))$ is a given Euclidean space.

Theorem 2.2.1. $\left(\sum_{i=1}^m a_i u_i, \sum_{j=1}^n b_j v_j\right) = \sum_{i=1}^m \sum_{j=1}^n \overline{a}_i b_j (u_i, v_j)$, *and* $(\overline{0}, u) = (u, \overline{0}) = 0$ *for all* $u \in V$.

Proof. Easy exercise. □

Def. If $u, v \in V$, u and v are said to be *orthogonal* provided $(u, v) = 0$. A set D of *nonzero* vectors in V is called an *orthogonal set* provided $(u, v) = 0$ whenever $u, v \in D$, $u \neq v$.

Def. For $u \in V$, the *norm* of u (relative to the inner product $(\ ,\ )$) is defined by:

$$\|u\| := (u, u)^{1/2}.$$

Exercise 2.2.1.1. $\|cu\| = |c| \cdot \|u\|$ *for all* $c \in F$, $u \in V$.

Def. An orthogonal set D is called *orthonormal* provided $\|u\| = 1$ for all $u \in D$.

Theorem 2.2.2. *An orthogonal set is linearly independent.*

Proof. Easy exercise. □

Let D be a nonempty orthogonal set and let $u \in V$. Suppose u is in the linear span $\langle D \rangle$ of D, say $u = \sum_{i=1}^m a_i u_i$, $u_i \in D$. Then $(u_j, u) = \sum_i a_i (u_j, u_i) = a_j (u_j, u_j)$, implying $a_j = (u_j, u)/(u_j, u_j)$. In general, even if we do not know whether $u \in \langle D \rangle$, we define the *component of* u *along* u_j to be

$$a_j = \frac{(u_j, u)}{(u_j, u_j)}, \quad \text{for each } u_j \in D.$$

Now suppose that D is a nonempty orthogonal set (finite or countably infinite): $D = \{u_1, u_2, \cdots\}$. Let $u \in V$ with $a_j = (u_j, u)/(u_j, u_j)$ the component of u along u_j.

Theorem 2.2.3. *(Parseval). For each finite integer N, $1 \leq N \leq |D|$, we have $0 \leq \|u - \sum_{i=1}^{N} a_i u_i\|^2 = \|u\|^2 - \sum_{i=1}^{N} |a_i|^2 \|u_i\|^2 \leq \|u\|^2$, with equality if and only if $a_i = 0$ for all i.*

Proof.

$$(u - \sum a_i u_i, u - \sum a_i u_i) = (u,u) - \sum a_i (u, u_i) - \sum \overline{a}_i (u_i, u) + \sum \overline{a}_i a_j (u_i, u_j)$$

$$= \|u^2\| - \sum \frac{(u_i, u)}{(u_i, u_i)} \cdot \frac{(u, u_i)}{(u_i, u_i)} (u_i, u_i) - \sum \frac{(u, u_i)(u_i, u)}{(u_i, u_i)(u_i, u_i)} (u_i, u_i) + \sum |a_i|^2 (u_i, u_i)$$

$$= \|u\|^2 - 2 \sum |a_i|^2 (u_i, u_i) + \sum |a_i|^2 (u_i, u_i)$$

$$= \|u\|^2 - \sum_{i=1}^{N} |a_i|^2 \|u_i\|^2.$$

□

There are two immediate corollaries.

Corollary 2.2.4. *If $u = \sum_1^r a_i u_i$, then $\|u\|^2 = \sum_1^r |a_i|^2 \|u_i\|^2$.*

Corollary 2.2.5. *(Bessel) For each finite integer N, $1 \leq N \leq |D|$, $\sum_1^N |a_i|^2 \cdot \|u_i\|^2 \leq \|u\|^2$, with equality if and only if $u = \sum a_i u_i$.*

Suppose D is infinite and orthonormal. Put $S_N = \sum_1^N a_i u_i$. By definition we say $u = lim_{N \to \infty} S_N = \sum_1^\infty a_i u_i$ provided $\lim_{N \to \infty} \|u - S_N\| = 0$.

Theorem 2.2.6. *Supppose that D is infinite and orthonormal and let S_N be as given above. Then we have the following.*

(i) (Bessel) $\sum_1^\infty |a_i|^2$ converges and $\sum_1^\infty |a_i|^2 \leq \|u\|^2$.

(ii) (Parseval) $\|u - S_N\|$ is monotonic decreasing as $N \to \infty$, and $u = \lim_{N \to \infty} S_N$ (i.e., $u = \sum_1^\infty a_i u_i$) if and only if $\|u\| = (\sum_1^\infty |a_i|^2)^{1/2}$

Proof. By Theorem 2.2.3, $\sum_1^N |a_i|^2 \leq \|u\|$ for each $N = 1, 2, \cdots$, implying that $\sum_1^\infty |a_i|^2$ converges to a limit bounded above by $\|u\|^2$. Also, by Theorem 2.2.3, $\|u - S_N\|^2 = \|u\|^2 - \sum_1^N |a_i|^2$ is clearly monotonic decreasing, with $lim_{N \to \infty} \|u - S_N\| = 0$ if and only if $\|u\|^2 = \sum_1^\infty |a_i|^2$. By definition we say $u = \lim_{N \to \infty} S_N = \sum_1^\infty a_i u_i$ provided $lim_{N \to \infty} \|u - S_N\| = 0$, so the proof is complete. □

Theorem 2.2.7. *(Schwarz Inequality)* $|(u,v)| \leq \|u\| \cdot \|v\|$, *with equality if and only if* $\{u,v\}$ *is a linearly dependent set.*

Proof. If $u = 0$ there is nothing to prove. So assume $u \neq 0$. Then $\{u\}$ is an orthogonal set. The component of v along u is $a = (u,v)/(u,u)$. And Cor 2.2.4 says $|a|^2 \cdot \|u\|^2 \leq \|v\|^2$, i.e., $[(u,v)(v,u)/(u,u)(u,u)](u,u) \leq (v,v)$, with equality if and only if $v = [(u,v)/(u,u)]u$. □

Corollary 2.2.8. *(Triangle Inequality)* $\|u+v\| \leq \|u\| + \|v\|$, *with equality if and only if* $v = cu$, $c > 0$ *(assuming* $u \neq 0$*).*

Proof. $\|u+v\|^2 = (u+v, u+v) = \|u\|^2 + |v\|^2 + (u,v) + \overline{(u,v)} = \|u\|^2 + \|v\|^2 + 2\mathcal{R}(u,v) \leq \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\| = (\|u\| + \|v\|)^2$, where the inequality holds by the Schwarz Inequality, with equality if and only if $\mathcal{R}(u,v) = \|u\| \cdot \|v\|$ if and only if $v = cu$ with $c > 0$. □

Theorem 2.2.9. *Let* D *be orthogonal and let* $u_1, \cdots, u_r$ *be finitely many elements of* D*. For* $u \in V$ *and* $a_i = (u_i, u)/(u_i, u_i)$*, and with* $v = \sum_1^r c_i u_i$*, we have*

(i) $(v,u) = (\sum_1^r c_i u_i, u) = \sum_1^r \overline{c}_i (u_i, u) = \sum_1^r \overline{c}_i a_i (u_i, u_i)$.

(ii) $(u_n, v) = \begin{cases} c_n(u_n, u_n) & \text{if } n \leq r; \\ 0 & \text{if } n > r. \end{cases}$

(iii) (Minimum distance property) $\|u - \sum_1^r a_i u_i\| \leq \|u - \sum_1^r c_i u_i\|$, *with equality if and only if* $a_i = c_i$, $i = 1, \ldots, r$. *(That is,* $\sum_1^r a_i u_i$ *is the unique vector* $\in \langle u_1, \ldots, u_r \rangle$ *closest to* u*.)*

Proof. Clearly (i) and (ii) are true. For (iii) ,

$$\begin{aligned}
\|u - \sum c_i u_i\|^2 &= \|u\|^2 - \sum \overline{c}_i (u_i, u) - \sum c_i (u, u_i) + \textstyle\sum_{i,j} \overline{c}_i c_j (u_i, u_j) \\
&= \|u\|^2 - \sum \overline{c}_i a_i (u_i, u_i) - \sum c_i \overline{a}_i (u_i, u_i) + \sum |c_i|^2 (u_i, u_i) \\
&= \|u\|^2 - \sum |a_i|^2 (u_i, u_i) + \sum (a_i - c_i)\overline{(a_i - c_i)}(u_i, u_i) \\
&= \|u\|^2 - \sum |a_i|^2 (u_i, u_i) + \sum |a_i - c_i|^2 (u_i, u_i) \\
&\geq \|u\|^2 - \sum |a_i|^2 (u_i, u_i),
\end{aligned}$$

with equality if and only if $a_i = c_i$, $1 \leq i \leq r$. □

Theorem 2.2.10. *(Gram-Schmidt Method) Let* $D = \{u_1, \ldots, u_r\}$ *be a finite orthogonal set with* $u \notin \langle D \rangle$*. Put* $u_{r+1} = u - \sum_1^r a_i u_i$*, with* $a_i = (u_i, u)/(u_i, u_i)$*, as usual. For* $1 \leq i \leq r$*,* $(u_i, u_{r+1}) = (u_i, u - \sum_1^r a_i u_i) = (u_i, u) - a_i (u_i, u_i) = 0$*. Hence* $D' = \{u_1, \ldots, u_r, u_{r+1}\}$ *is also orthogonal and*

it is easy to check that $D' = \langle D \cup \{u\}\rangle$. *This is the basis of the following method.*

Suppose $\mathcal{B} = \{\beta_1, \beta_2, \cdots\}$ *(finite or infinite) is an independent set. Put* $u_1 = \beta_1, u_2 = \beta_2 - [(u_1, \beta_2)/(u_1, u_1)]u_1, \ldots, u_r = \beta_r - \sum_1^{r-1}[(u_i, \beta_r)/(u_i, u_i)]u_i$. *It follows that* $D = \{u_1, \ldots, u_r\}$ *is an orthogonal set for each* r, $1 \le r \le |\mathcal{B}|$. *And for each* i, $1 \le i \le r$, $\langle u_1, \ldots, u_i\rangle = \langle \beta_1, \ldots, \beta_i\rangle$.

Exercise 2.2.10.1. *Let* V *be an Euclidean space with inner product* $(\ ,\ )$ *with norm* $\|\cdot\|$ *defined as usual using the inner product. Prove the following:*

(i) Parallelogram Equality: $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$.

(ii) Real Polarization Identity: $(u, v) = \frac{1}{4}\left\{\|u+v\|^2 - \|u-v\|^2\right\}$.

(iii) Complex Polar Identity: $4(u, v) = \|u+v\|^2 - \|u-v\|^2 + i\left\{\|u+iv\|^2 - \|u-iv\|^2\right\}$.

2.3 Euclidean and Metric Spaces

A set M together with a "distance" function $d : M \times M \to \mathcal{R}$ is a *metric space* provided the following hold:

1. $d(p, q) \ge 0$ for all $p, q \in M$, with equality if and only if $p = q$.
2. $d(p, q) = d(q, p)$ for all $p, q \in M$.
3. $d(p, q) \le d(p, r) + d(r, q)$ for all $p, q, r \in M$ ("triangle property").

If V is a euclidean space with inner product $(,)$, define $d(p, q) = \|p - q\| = \sqrt{(p-q, p-q)}$. Then the triangle property is just the usual triangle inequality:

$$d(p, q) = \|p - r + r - q\| \le \|p - r\| + \|r - q\| = d(p, r) + d(r, q).$$

Example 2.3.0.2. *In* $\mathcal{R}^n$, $((a_i), (b_i)) = \sum a_i b_i$ *is the usual dot product with* $d((a_i), (b_i)) = \sqrt{\sum (a_i - b_i)^2}$.

Example 2.3.0.3. *In* $\mathcal{C}^n$, $((a_i), (b_i)) = \sum \overline{a}_i b_i$ *is the usual hermitian inner product with* $d((a_i), (b_i)) = \sqrt{\sum |a_i - b_i|^2}$.

Example 2.3.0.4. *In* $C[a, b]$ *put* $(f, g) = \int_a^b f(t)g(t)dt$. *Then*

$$d(f, g) = \sqrt{\int_a^b [f(t) - g(t)]^2 dt}$$

(= the "mean square deviation" of the functions f *and* g*).*

Example 2.3.0.5. *Let $w \in C[a,b]$ with $w(t) > 0$ for all $t \in [a,b]$. For $f, g \in \mathcal{C}[a,b]$, put $(f,g) = \int_a^b w(t)f(t)g(t)dt$, then $d(f,g) = \sqrt{\int_a^b w(t)(f(t)-g(t))^2 dt}$.*

Example 2.3.0.6. *Let $P(x)$ be the space of all real polynomials in x. Put $(f,g) = \int_0^\infty e^{-t}f(t)g(t)dt$, so $d(f,g) = \sqrt{\int_0^\infty e^{-t}(f(t)-g(t))^2 dt}$.*

Example 2.3.0.7. *Let $\mathcal{H} = \{f : [a,b] \to \mathcal{C} : f$ is continuous$\}$. So a typical $f \in \mathcal{H}$ looks like $f = u + iv$, where $u, v \in C[a,b]$. Put $(f,g) = \int_a^b \overline{f}(t)g(t)dt$. Then $d(f,g) = \sqrt{\int_a^b |f(t)-g(t)|^2 dt}$.*

Example 2.3.0.8. *$l_2 = \{p = (a_1, a_2, \ldots) : \sum_1^\infty a_i^2$ converges$\}$. Put $(p,q) = \sum_1^\infty a_i b_i$, and $d(p,q) = \left(\sum_1^\infty (a_i - b_i)^2\right)^{1/2}$, where $p = (a_1, a_2, \ldots)$, $q = (b_1, b_2, \ldots)$, $a_i,\ b_i \in \mathcal{R}$. The space l_2 with this inner product is real (separable) Hilbert space, and we will study it more a little later.*

Example 2.3.0.9. *In $C[a,b]$ put $d(f,g) = \|f-g\|_{[a,b]} = max\{|f(t)-g(t)| : a \le t \le b\}$. This gives a metric, but it did not come from an inner product.*

In a metric space (M,d), the sequence $\{p_n\}$ *converges to* p (written $\{p_n\} \to p$) provided $lim_{n\to\infty} d(p_n, p) = 0$. Given $\sum_1^\infty u_i$, $u_i \in M$, we say $\sum_1^\infty u_i$ *converges to* u and write $\sum_1^\infty u_i \to u$ or $u = \sum_1^\infty u_i$ provided $\{p_n\} \to u$, where $p_n = \sum_1^n u_i$. (In order for this to make sense, M must have an additive structure. Usually M is a vector space over some subfield of $\mathcal{C}$.)

It follows that in $C[a,b]$ we have three kinds of convergence for sequences and hence for series also.

Pointwise convergence: $\{f_n\} \to f$ (pointwise) if and only if for each $t \in [a,b]$, $lim_{n\to\infty} f_n(t) = f(t)$.

Convergence in the mean: $\{f_n\}^* \to f$ (or $f = l.i.m.\{f)_n\}$) provided $lim_{n\to\infty}{}^*\|f_n - f\| = 0$ if and only if $lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$. (Write ${}^*lim_{n\to\infty} f_n = f$.)

Uniform convergence: $\{f_n\} \to f$ *uniformly on* $[a,b]$ if and only if $lim_{n\to\infty}\|f_n - f\|_{[a,b]} = 0$. (See Example 2.3.0.9.)

Theorem 2.3.1. *In $C[a,b]$, if $\{f_n\} \to f$ uniformly, then the sequence converges to f both pointwise and in the mean.*

Proof. It is clear that uniform convergence on an interval $[a, b]$ implies pointwise convergence at each point of $[a, b]$. Moreover, convergence in the mean is an immediate consequence of the following inequality:

$$\begin{aligned} {}^*\|f-g\| &= \left\{\int_a^b |f(t)-g(t)|^2 dt\right\}^{1/2} \\ &\le \left\{\int_a^b \|f-g\|_{[a,b]}^2 dt\right\}^{1/2} \\ &= \sqrt{b-a}\cdot\|f-g\|_{[a,b]}. \end{aligned}$$

□

As an example to show that the two norms are really different, put $f(t) = t^2$, $g(t) = t^3$, $I = [0, 1]$. Then

$$\begin{aligned} {}^*\|f-g\| &= \left(\int_0^1 |t^2-t^3|^2 dt\right)^{1/2} \\ &= \left(\int_0^1 (t^4-2t^5+t^6) dt\right)^{1/2} \\ &= \left(\tfrac{1}{5}-\tfrac{1}{3}+\tfrac{1}{7}\right)^{1/2} \\ &= \tfrac{1}{\sqrt{105}}. \end{aligned}$$

On the other hand, $\|f-g\|_I = \frac{4}{27}$ (after a little computation). However, Theorem 2.3.1 shows that it is not possible for $\{f_n\}$ (with $f_n \in C[a, b]$) to converge in the mean to f and converge uniformly to g, with $f \neq g$.

A more interesting example shows that a sequence $\{f_n\}$ of functions in $C[a, b]$ may converge to a function f both in the mean and pointwise but not uniformly. Put $f_n(t) = n\sqrt{t}e^{-n^2 t}$, $t \in [0, 1]$. Then ${}^*\lim_{n\to\infty}\|f_n(t)\| = \overline{0}$, since

$$\begin{aligned} \int_0^1 |f_n(t)|^2 dt &= \int_0^1 n^2 t e^{-2n^2 t} dt \\ &= \tfrac{1}{n^2}\int_0^1 (n^2 t) e^{-2n^2 t}\cdot(n^2 dt) \\ &= \tfrac{1}{n^2}\int_0^{n^2} s\ e^{-2s} ds \\ &= \tfrac{1}{n^2}\left[-\tfrac{1}{2}s\ e^{-2s} - \tfrac{1}{4}e^{-2s}\right]_0^{n^2} \\ &= \tfrac{1}{n^2}\left[\left(\tfrac{-n^2}{2}e^{-2n^2} - \tfrac{1}{4}e^{-2n^2}\right) - (-0-\tfrac{1}{4})\right] \to 0 \end{aligned}$$

as $n \to \infty$. But $\{f_n\}$ does not converge to $\overline{0}$ uniformly on $I = [0, 1]$, since $\|f_n\|_I = \max_{0\le t\le 1}|f_n(t)| = |f_n(\frac{1}{2n^2})| = \frac{1}{\sqrt{2e}}$ for all n. (To obtain this, set the derivative equal to 0.) On the other hand, $lim_{n\to\infty}\frac{n\sqrt{t}}{e^{n^2 t}} = \lim_{n\to\infty}\frac{\sqrt{t}}{2nte^{n^2 t}} = 0$ for each $t \in (0, 1]$. And $\lim_{n\to\infty} f_n(0) = 0$ also. So $\{f_n\} \to \overline{0}$ pointwise.

To show that convergence in the mean does not imply pointwise convergence, put

$$g_n = \begin{cases} 0, \text{ if } 0 \le x \le \frac{1}{2} - \frac{1}{2^n}, \text{ or } \frac{1}{2} + \frac{1}{2^n} \le x \le 1; \\ 2^n x + 1 - 2^{n-1}, \text{ if } \frac{1}{2} - \frac{1}{2^n} \le x \le \frac{1}{2}; \\ -2^n x + 1 + 2^{n-1}, \text{ if } \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{2^n}. \end{cases}$$

Then $\{g_n\}$ does not converge pointwise to a function of $C[0,1]$, but $lim_{n\to\infty}{}^*\|g_n - \overline{0}\| = 0$. (The sequence $\{g_n\}$ converges pointwise to the function whose value at $1/2$ is 1, and whose value at all other points of $[0,1]$ is 0.)

As a last example, let $h_n(t)$ be the nonnegative function on $[0,1]$ for which the graph of $(h_n(t))^2$ has the following shape: the line segment from $(0,0)$ to $(\frac{1}{2^{n+1}}, 2^{n+1})$, the line segment from $(\frac{1}{2^{n+1}}, 2^{n+1})$ to $(\frac{1}{2^n}, 0)$, and the line segment from $(\frac{1}{2^n}, 0)$ to $(1,0)$. It follows that $\left(\int_0^1 [h_n(t) - \overline{0}]^2 dt\right)^{1/2} = 1$. So $\{h_n\} \to \overline{0}$ pointwise, but not in the mean.

It follows that of the six possible implications between the pairs of convergence types, the only two that are valid are given by Theorem 2.3.1.

2.4 Fourier Coefficients

Return to the setting of a general Euclidean space V with orthonormal set $D = \{u_i : i = 1, 2, 3, \ldots\}$, but from now on assume that $F = \mathcal{R}$. For $u \in V$ compute $a_n = (u_n, u)/(u_n, u_n) = (u_n, u)$, $n = 1, 2, \cdots$. The sequence $\{a_n\}$ is called the sequence of *Fourier coefficients* of u with respect to the orthonormal set D, and we write $u \sim \{a_n\}$ or $u \sim \sum_1^\infty a_n u_n$ to denote this connection between u and its Fourier coefficients. But it is important to keep in mind that $\sum_1^\infty a_n u_n$ may not converge, and even if it does converge to some $w \in V$, perhaps $w \ne u$.

Theorem 2.4.1. *Let $\{a_n\}_1^\infty \subseteq \mathcal{R}$ with $\sum_1^\infty a_n u_n \to u \in V$. Then*

(i) $\sum_1^\infty |a_n|^2 < \infty$.

(ii) $u \sim \{a_n\}_{n=1}^\infty$.

(iii) For $v \in V$, if $v \sim \{c_n\}$, then $(u, v) = \sum_{n=1}^\infty a_n c_n$.

Proof. Put $S_N = \sum_1^N a_n u_n$. By hypothesis, $lim_{N\to\infty} S_N = u$. By Theorem 2.2.9, for any $v \in V$, $(v, S_N) = \sum_1^N c_n a_n$, so that $(v, u) - \sum_1^N c_n a_n = (v, u) - (v, S_N) = (v, u - S_N)$. Hence $|(v, u) - \sum_1^N c_n a_n| = |(v, u - S_N)| \le$

$\|v\| \cdot \|u - S_N\|$. Here $\|v\|$ is fixed and $\|u - S_N\|$ goes to 0 as $N \to \infty$, by hypothesis. This proves (iii).

For (ii), put $v = u_n$ in (iii) to get $(u_n, u) = 1 \cdot a_n$. Finally, for (i), we know that $a_n = (u_n, u)/(u_n, u_n)$, so $\sum |a_n|^2 < \infty$ by Theorem 2.2.6 part (i). □

Exercise 2.4.1.1. *If $\{x_n\} \to x$ and $\{y_n\} \to y$, then $(x_n, y_n) \to (x, y)$.*

Proof. It is simple:

$$\begin{aligned} |(x,y) - (x_n, y_n)| &= |(x, y - y_n) + (y, x - x_n) - (x - x_n, y - y_n)| \\ &\le |(x, y - y_n)| + |(y, x - x_n)| + |(x - x_n, y - y_n)| \\ &\le \|x\| \cdot \|y - y_n\| + \|y\| \cdot \|x - x_n\| + \|x - x_n\| \cdot \|y - y_n\|, \end{aligned}$$

which approaches 0 as $n \to \infty$. □

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ in any Euclidean space V is said to be a *Cauchy sequence* provided that for any $\epsilon > 0$ there is an N such that $\|x_n - x_m\| < \epsilon$ for all $m, n \ge N$.

Theorem 2.4.2. *Every convergent sequence is Cauchy.*

Proof. Assume $\{x_n\} \to x$, i.e., $lim_{n\to\infty}\|x_n - x\| = 0$. then $\|x_n - x_m\| \le \|x_n - x\| + \|x - x_m\|$, from which the result follows easily. □

Theorem 2.4.3. *Every Cauchy sequence is bounded.*

Proof. Supose that $\{x_n\}$ is Cauchy and let $\epsilon = 1$. Then choose N such that $\|x_n - x_m\| < 1$ whenever $m, n \ge N$. Then for $m \ge N$, $\|x_m\| \le \|x_m - x_N\| + \|x_N\| < 1 + \|x_N\|$. Put $M = max\{\|x_1\|, \cdots, \|x_{N-1}\|, \|x_N\| + 1\}$. Then $\|x_m\| \le M$ for all m. □

If $\sum a_n u_n$ converges, we know it is Cauchy and $\sum |a_n|^2$ converges. There is a partial converse.

Theorem 2.4.4. *If $\sum_1^\infty |a_n|^2 < \infty$, then $\sum_1^\infty a_n u_n$ is Cauchy.*

Proof. Put $S_N = \sum_1^N a_n u_n$. For $M \ge N$, by Corollary 2.2.4, $\|S_M - S_N\|^2 = \sum_{i=N+1}^M |a_i|^2$. Since $\sum |a_i|^2 < \infty$, clearly $lim_{M,N\to\infty}\|S_M - S_N\|^2 = 0$, implying $\{S_N\}$ is Cauchy. □

Unfortunately, in general there is no reason to believe that V is *complete*, i.e., that each Cauchy sequence of points in V converges to a point of V.

Exercise 2.4.4.1. *Let $^*C(a, b)$ be the space $C(a, b)$ with inner product $(f, g) = \int_a^b f(t)g(t)dt$. Show that $^*C(a, b)$ is not complete.*

Proof. We construct a sequence $\{f_n\}$ in ${}^*C(-1,1)$ which satisfies the Cauchy criterion $\lim_{m,n\to\infty} {}^*\|f_n - f_m\| = 0$ but which does not converge to a function in ${}^*C(-1,1)$. Put $f_n(t) = t^{1/(2n-1)}$, $-1 \le t \le 1$, $n = 1, 2, \ldots$. With the metric $d(f,g) = {}^*\|f-g\| = \{\int_{-1}^{1} |f(t)-g(t)|^2 dt\}^{1/2}$, the given sequence converges to the function g given by

$$g(t) = \begin{cases} 1, & t > 0; \\ 0, & t = 0; \\ -1, & t < 0. \end{cases}$$

But g is not continuous, so $g \notin {}^*C(-1,1)$. □

2.5 Real Separable Hilbert Space

Recall that $\ell_2 = \{p = (a_1, a_2, \cdots) : a_i \in \mathcal{R} \text{ and } \sum_1^\infty a_i^2 < \infty\}$. Vector addition and scalar multiplication are "coordinatewise" and give a vector space over $\mathcal{R}$. For $p = (a_i)$, $q = (b_i)$, the inner product is $(p,q) = \sum_1^\infty a_i b_i$. Since

$$\left|\sum_{k=M}^{N+M} a_k b_k\right| \le \sqrt{\sum_{k=M}^{N+M} a_k^2} \cdot \sqrt{\sum_{k=M}^{N+M} b_k^2},$$

it follows from the Cauch criterion that the sum (p,q) really does converge. And $\sum_{k=M}^{N+M} (ca_k)^2 = c^2 \sum_{k=M}^{N+M} a_k^2$, so that $\sum_1^\infty (ca_k)^2$ converges. Hence the scalar product converges. Moreover, since

$$\left|\sum_{k=M}^{N+M} (a_k + b_k)^2\right| \le \left|\sum_{k=M}^{N+M} a_k^2\right| + 2\left|\sum_{k=M}^{N+M} a_k b_k\right| + \left|\sum_{k=M}^{N+M} b_k^2\right|,$$

it follows again from the Cauchy criterion that $\sum (a_k + b_k)^2$ converges. Hence vector addition in ℓ_2 is well-defined.

Theorem 2.5.1. *ℓ_2 is complete.*

Proof. Let $p_m = \{a_n^{(m)}\}_{n=1}^\infty$, for $m = 1, 2, \cdots$. And suppose that $\{p_m\}$ is a Cauchy sequence in ℓ_2. By hypothesis, $\lim_{m,k\to\infty} \|p_m - p_k\|^2 =$ $\lim_{m,k\to\infty} \left(\sum_{m=1}^\infty |a_n^{(m)} - a_n^{(k)}|^2\right) = 0$. So, in particular for fixed n, $\lim_{m,k\to\infty} |a_n^{(m)} - a_n^{(k)}|^2 = 0$, implying $\lim_{m,k\to\infty} |a_n^{(m)} - a_n^{(k)}| = 0$. Hence, for fixed n, the sequence $\{a_n^{(m)}\}_{m=1}^\infty$ converges by the Cauchy criterion. Write

$a_n = \lim_{m\to\infty} a_n^{(m)}$. We shall show that $p = (a_1, a_2, \cdots) \in \ell_2$ and that $\{p_m\} \to p$.

First, since a Cauchy sequence is bounded, there is a K (not depending on m) such that $\|p_m\|^2 = \sum_{n=1}^{\infty} |a_n^{(m)}|^2 \le K$. Then for fixed N, $\sum_{n=1}^{N} a_n^2 = \lim_{m\to\infty} \sum_{n=1}^{N} |a_n^{(m)}|^2 \le K$, which implies that $\sum_{n=1}^{\infty} a_n^2$ is convergent, so that $p \in \ell_2$.

To see that $\{p_m\} \to p$, let $\epsilon > 0$ be given. Then there is an N (depending only on ϵ) such that $\|p_m - p_k\| < \epsilon$ for all $m, k \ge N$, i.e.,

$$\sqrt{\sum_{n=1}^{\infty} (a_n^{(m)} - a_n^{(k)})^2} < \epsilon \ \forall\ m, k \ge N.$$

It follows that for each integer $T = 1, 2, \cdots$

$$\sqrt{\sum_{n=1}^{T} (a_n^{(m)} - a_n^{(k)})^2} < e \ \forall m, k \ge N.$$

Hence

$$\lim_{k\to\infty} \sqrt{\sum_{n=1}^{T} (a_n^{(m)} - a_n^{(k)})^2} = \sqrt{\sum_{n=1}^{T} (a_n^{(m)} - a_n)^2} \le \epsilon, \text{ for } m \ge N.$$

Then taking the limit as $T \to \infty$, we have

$$\sqrt{\sum_{n=1}^{\infty} (a_n^{(m)} - a_n)^2} = \|p_m - p\| \le \epsilon \ \forall m \ge N,$$

implying $\{p_m\} \to p$. □

Next we show that ℓ_2 has a "standard" basis: put $e_n = (0, 0, \cdots, 0, 1, 0, \cdots)$, $n = 1, 2, \cdots$.

By saying that $\Sigma = \{e_1, e_2, \cdots\}$ is a "basis", we mean that for $p \in \ell_2$ there are unique $c_i \in \mathcal{R}$, $i = 1, 2, \cdots$, for which $p = \sum_1^{\infty} c_i e_i$. Since $c_n = (p, e_n) = (e_n, p)/(e_n, e_n)$ ($= a_n$ with earlier notation), the c_i are uniquely determined.

If $S_N = \sum_1^N c_i e_i$, we must show that $S_N \to p$, i.e., $\lim_{N\to\infty} \|p - S_N\| = 0$. But $\|p - S_N\|^2 = \sum_{k=N+1}^{\infty} |c_k|^2$. And since $p \sim \{c_k\}$, by Theorem 2.2.6 (i) we know that $\sum_1^{\infty} |c_k|^2$ converges, so that $\lim_{N\to\infty} \sum_{k=N+1}^{\infty} |c_k|^2 = 0$.

2.6 The Space $^*C(a,b)$ (Example 2.3.0.4)

Recall that $^*C(a,b)$ is the set of all continuous functions on $[a,b]$ with inner product $(f,g) = \int_a^b f(t)g(t)dt$, and distance function $d(f,g) = {}^*\|f-g\| = \left\{\int_a^b |f(t)-g(t)|^2 dt\right\}^{1/2}$. Write $^*\lim_{n\to\infty} f = g$ provided $\lim_{n\to\infty} {}^*\|f-g\| = 0 \iff \lim_{n\to\infty}\int_a^b |f_n(t)-g(t)|^2 dt = 0$. and write $^*C$ for $^*C(a,b)$.

An orthonormal set $\{\phi_n\}_{n=1}^\infty$ in $^*C$ is a *Fourier basis* for $^*C$ provided each $f \in {}^*C$ has the unique expansion $f = \sum_1^\infty c_n\phi_n$, where $c_n = (f,\phi_n)$.

Let $\{\phi_n\}_{n=1}^\infty$ be a given orthonormal set. For any $f \in {}^*C$, compute $c_n = (f,\phi_n)$ and form the formal series $\sum_1^\infty c_n\phi_n$. Write $f \sim \sum_1^\infty c_n\phi_n$ in this case. Note: $(f,\phi_n) = \int_a^b f(t)\phi_n(t)dt$. Keep in mind that $\{\phi_n\}_{n=1}^\infty$ might not be "complete," i.e., a large enough set to be a basis. If $f = \sum_1^\infty c_n\phi_n$, we know that $f \sim c_n\phi_n$. but if $f \sim \sum_1^\infty c_n\phi_n$, it may not be the case that $f = \sum_1^\infty c_n\phi_n$.

The example of Fourier (around 1820), and the orthonormal set most important historically in physics, is the following:

$$(*)\quad \phi_0 = \frac{1}{\sqrt{2\pi}},\ \phi_{2k}(x) = \frac{cos\ kx}{\sqrt{\pi}},\ \phi_{2k-1}(x) = \frac{sin\ kx}{\sqrt{\pi}},\ k = 1,2,\cdots,$$

and the standard interval is $[a,b] = [-\pi,\pi]$, although $[0,2\pi]$ or even $[a,a+2\pi]$ works just as well for these functions. From now on we refer to this set $(*)$ as the trigonometric set $(*)$.

Exercise 2.6.0.1. *Show that the trignonometric set $(*)$ is an orthonormal set. (Hint: Review Lemma 1.5.1.)*

In general we let A_0 denote the Fourier coefficient along $\frac{1}{\sqrt{2\pi}}$, and for $n \geq 1$ we let A_n be the Fourier coefficient along $\frac{\cos(nx)}{\sqrt{\pi}}$, and B_n be the Fourier coefficient along $\frac{\sin(nx)}{\sqrt{\pi}}$.

Remark 2.6.1. *Historically, the most commonly used orthogonal set has been $\{u_0, u_1, \cdots\}$ where $u_0 = \frac{1}{2}$, $u_{2k} = cos(kx)$, $u_{2k-1} = sin(kx)$, $k = 1,2,\cdots$. The component of f along u_i is $a_i = (u_i,f)/(u_i,u_i)$. This gives:*

$$\begin{aligned} f &\sim \frac{1}{2}a_0 + \sum_1^\infty (a_n cos(nx) + b_n sin(nx)) \\ &= A_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^\infty \left(A_n \frac{cos(nx)}{\sqrt{\pi}} + B_n \frac{sin(nx)}{\sqrt{\pi}} \right), \end{aligned}$$

so $A_0 = \frac{\sqrt{\pi}}{\sqrt{2}} a_0$, *and* $A_n = a_n\sqrt{\pi}$, $B_n = b_n\sqrt{\pi}$, *for* $n \geq 1$. *So by Theorem 2.2.6, if* f *is continuous on the interval* $[-\pi, \pi]$, *then*

$$f = l.i.m \left\{ \frac{1}{2}a_0 + \sum_1^\infty (a_n cos(nx) + b_n sin(nx)) \right\}$$

if and only if

$$|A_0|^2 + \sum_1^\infty |A_n|^2 + \sum_1^\infty |B_n|^2 = \|f\|^2$$

if and only if

$$\frac{a_0^2}{2} + \sum |a_n|^2 + \sum |b_n|^2 = \frac{1}{\pi}\|f\|^2$$

(Parseval's equality).

Example 2.6.1.1. $f(x) = \frac{x}{2}$. *An easy computation shows that* $a_n = 0$ *for* $n = 0, 1, \cdots$, *and* $b_n = \frac{(-1)^{n+1}}{n}$. *Hence* $\frac{x}{2} \sim \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} sin(nx)$. *We can use the Dirichlet test to show that the series is pointwise convergent for every* $x \in [-\pi, \pi]$. *Put* $a_n = \frac{(-1)^{n+1}}{n}$ *and* $b_n = sin(nx)$ *in the notation of the Dirichlet test. Then* $a_n \to 0$, $\sum |a_{n+1} - a_n| < \infty$, *so* $\sum a_n b_n$ *converges if the partial sums of* $\sum b_n$ *are bounded. By Eq. 1.13 we know that*

$$\sum_1^n sin(kx) = \frac{cos\left(\frac{x}{2}\right) - cos\left(n + \frac{1}{2}\right)x}{2sin\left(\frac{x}{2}\right)} \leq \frac{1}{\sin(x/2)},$$

for all x *with* $sin\left(\frac{x}{2}\right) \neq 0$.

Hence the original series converges for all $x \in [-\pi, \pi]$ *(actually, for all* $x \in \mathcal{R}$*) for which* $sin\left(\frac{x}{2}\right) \neq 0$. *For* $x = 0$ *the series clearly converges.*

For $x = \pm\pi$, however, $\frac{x}{2} \neq 0 = \sum \frac{(-1)^{"n+1}}{n} \cdot 0$. So Parseval's relation becomes

$$\begin{aligned} 0 + \sum 0 + \sum \left|\frac{1}{n}\right|^2 &= \frac{1}{\pi} \cdot \|\frac{x}{2}\|^2 \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4} dx \\ &= \frac{1}{\pi}\left[\frac{x^3}{12}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^3 + \pi^3}{12}\right) \\ &= \frac{\pi^2}{6}. \end{aligned}$$

It is well known that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (and we will establish this later independently), so that we now know that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx) \ ^* = \ (in\ the\ mean)\ \ \frac{x}{2}.$$

Hence here is another example to show that convergence in the mean is not the same as pointwise convergence.

For easy reference we review the formulas for the a_n and b_n.

$$a_0 = \frac{\sqrt{2}}{\sqrt{\pi}} A_0 = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{f(t)}{\sqrt{2\pi}} dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) dt. \tag{2.1}$$

For $n \geq 1$

$$a_n = \frac{A_n}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(t) \frac{\cos(nt)}{\sqrt{\pi}} dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt) dt, \tag{2.2}$$

and

$$b_n = \frac{B_n}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(t) \frac{\sin(nt)}{\sqrt{\pi}} dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt) dt. \tag{2.3}$$

Example 2.6.1.2. *$f(x) = \frac{x^2}{4} \sim \frac{\pi^2}{12} - cos(x) + \frac{1}{4} cos(2x) - \frac{1}{9} cos(3x) + \cdots$. Since $\sum \frac{1}{n^2}$ converges, by the M-test, the Fourier series for $\frac{x^2}{4}$ converges uniformly on $(-\infty, \infty)$ and hence uniformly on $[-\pi, \pi]$, which then implies convergence*

in the mean and pointwise. But if $g(x)$ is the function to which the series converges uniformly (and hence both in the mean and pointwise) on $[-\pi,\pi]$, we do not yet know if $\frac{x^2}{4}$ is the function to which it converges. This will follow, however, from the remaining results of this section.

We have seen that the space $^*C$ is not complete, i.e., not every Cauchy sequence converges to a function in $^*C$. This can be overcome by enlarging the space $^*C$ by including a great many discontinuous functions. When this is done, arriving at a function space $\mathcal{F}$ that is complete, then every series $\sum_1^\infty a_n\phi_n$ with $\sum_1^\infty |a_n|^2 < \infty$ is convergent, and is therefore the Fourier series of a member of $\mathcal{F}$. But this process requires a much more thorough discussion of integration and Lebesgue measure theory, and we say no more of it here.

The question we want to consider here is: How can one tell if an orthonormal set $\{\phi_n\}$ is a Fourier basis for the space $^*C$?

Theorem 2.6.2. *The orthonormal set $\{\phi_n\}$ is a basis for $^*C$ if and only if the collection of all Fourier polynomials is dense in $^*C$.*

Proof. A Fourier polynomial is just a finite linear combination of elements of $\{\phi_n\}$. By "dense," we mean that given any $f \in {}^*C$ and any $\epsilon > 0$ there is a Fourier polynomial P with $^*\|f-P\| < \epsilon$. This clearly holds if $\{\phi_n\}$ is a basis, since P could be chosen as one of the partial sums of the Fourier series for f, which by assumption converges to f. Conversely, suppose the polynomials are dense: We must then prove that the Fourier series for any f converges to f.

Choose any $\epsilon > 0$, and then P with $^*\|f-P\| < \epsilon$. Say $P = \sum_1^N a_n\phi_n$.

$$\begin{aligned} {}^*\|f-S_k\| &\leq {}^*\|f-S_N\| \text{ for } k>N, \text{ by Theorem 2.2.6(ii)} \\ &\leq {}^*\|f-P\| < \epsilon, \quad \text{by Theorem 2.2.9(iii)} \end{aligned}$$

Hence $^*lim_{k\to\infty}S_k = f$, and $f = {}^*\sum_1^\infty c_n\phi_n$. □

We now restrict our attention to the classical trigonometric Fourier series. Our next result shows that the Fourier coefficients of a function $f \in {}^*C$ characterize the function.

A function of the form

$$P(x) = \sum_{k=0}^{N}(\alpha_k\cos(kx) + \beta_k\sin(kx))$$

is called a *trigonometric polynomial.*

Lemma 2.6.3. *Each of the functions* $(\cos(x))^m$ *is a trigonometric polynomial.*

Proof. Recall that $(\cos(x))^2 = \frac{1}{2} + \left(\frac{1}{2}\right)\cos(2x)$, and proceed by induction. Suppose that

$$(\cos(x))^N = c_0 + c_1\cos(x) + c_2\cos(2x) + \cdots + c_N\cos(Nx).$$

Then using $2\cos A\cos B = \cos(A-B) + \cos(A+B)$,

$$\begin{aligned}(\cos(x))^{N+1} &= \sum_0^N c_k\cos(x)\cos(kx) \\ &= \sum_0^N \frac{1}{2}c_k[\cos((k+1)x) + \cos((k-1)x)].\end{aligned}$$

□

Recall Lemma 1.3.5 as follows.

Lemma 2.6.4. *If* $|x| \leq \pi$*, then* $1 - \frac{1}{2}x^2 \leq cos(x) \leq 1 - \frac{1}{5}x^2$.

Theorem 2.6.5. *The trigonometric functions* $(*)$ *form a maximal orthogonal set on the interval* $I = [-\pi, \pi]$.

Proof. What we must prove is that if f is continuous in the interval $I = [-\pi, \pi]$ and if $\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \int_{-\pi}^{\pi} f(x)\sin(nx) = 0$ for all $n = 0, 1, \cdots$, then f is identically 0 on I. Clearly the hypotheses force

$$(**) \int_{-\pi}^{\pi} f(x)P(x) = 0 \text{ for any trigonometric polynomial } P(x).$$

We use $(**)$ to show that $f(x) = 0$ for each $x \in I$ by choosing a very special polynomial $P(x)$, and we do this first for $x = 0$.

Suppose that $f(0) \neq 0$ and without loss of generality that that $f(0) > 0$. Since f is continuous there is an $\epsilon > 0$ and a neighborhood about 0 in I on which $f(x) > \epsilon$. We need $P(x)$ to be extremely small outside this neighborhood of 0, but very large for points x near 0. Such a function clearly cannot satisfy $(**)$.

So suppose $f(x) > \epsilon$ for all x with $|x| < \delta$. Then if $|x| < \delta/3$, by Lemma 2.6.4 $\frac{1}{6}\delta^2 + \cos(x) > \frac{1}{6}\delta^2 + 1 - \frac{1}{2}\left(\frac{1}{9}\delta^2\right) = 1 + \frac{1}{9}\delta^2$. And if $\delta < |x| \leq \pi$, $-1 + \frac{1}{6}\delta^2 \leq \frac{1}{6}\delta^2 + \cos(x) < \frac{1}{6}\delta^2 + 1 - \frac{1}{5}\delta^2 = 1 - \frac{1}{30}\delta^2$. Choose n very large, and set $P(x) = \left(\frac{1}{6}\delta^2 + \cos(x)\right)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} \left(\frac{1}{6}\delta^2\right)^{n-i} (\cos(x))^i$. By Lemma 2.6.3 $P(x)$ is a trigonometric polynomial. For $|x| < \delta/3$, $\left(\frac{1}{6}\delta^2 + \cos(x)\right)^{2n} > \left(1 + \frac{\delta^2}{9}\right)^{2n}$, so $P(x)$ is uniformly arbitrarily large.

For $\delta < |x| \leq \pi$, $\left(\frac{1}{6}\delta^2 + \cos(x)\right)^{2n} < \left(1 - \frac{1}{30}\delta^2\right)^{2n}$, so $P(x)$ is uniformly arbitrarily small. So $(**)$ must fail to hold. This shows $P(0) = 0$. A similar argument can be made at any point $x = c \in I$ by using $P(x) = \left(\frac{1}{6}\delta^2 + \cos(x-c)\right)^{2n}$, where $|P(x)| > \epsilon$ for all x with $|x - c| < \delta$. □

Lemma 2.6.6. *Let f have a continuous derivative on $[-\pi, \pi]$ and suppose that $f(\pi) = f(-\pi)$. Then the Fourier coefficients $\{a_n\}$, $\{b_n\}$ for f obey $\sum_0^\infty |a_n| < \infty$, $\sum_1^\infty |b_n| < \infty$.*

Proof. By the formulas for a_n, b_n we find the Fourier coefficients α_n, β_n of f' in terms of those of f.

$$\begin{aligned}
\alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)dx \\
&= \frac{1}{\pi} f(x)\cos(nx)\Big]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= (-1)^n(f(\pi) - f(-\pi)) + nb_n \\
&= 0 + nb_n.
\end{aligned}$$

$$\begin{aligned}
\beta_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)dx \\
&= \frac{1}{\pi} f(x)\sin(nx)\Big]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= 0 - na_n.
\end{aligned}$$

Since α_n and β_n are Fourier coefficients of a function in $^*C$, by Theorem 2.2.6(i) both $\sum |\alpha_n|^2$ and $\sum |\beta_n|^2$ converge. Hence $\sum n^2|a_n|^2$ and $\sum n^2|b_n|^2$ both converge. However, a simple argument with Schwarz's inequality shows that in general if $\sum n^2|c_n|^2$ converges, so must the series $\sum |c_n|$:

$$\begin{aligned}\left(\sum_1^N |c_n|\right)^2 &= \sum_1^N (n \cdot |c_n| \cdot \frac{1}{n})^2 \\ &\leq \sum_1^N n^2 |c_n|^2 \cdot \sum \frac{1}{n^2} \\ &\leq \sum_1^\infty n^2 |c_n|^2 \cdot \sum_1^\infty \frac{1}{n^2},\end{aligned}$$

for all N. Applying this separately, we have $\sum |a_n|$ and $\sum |b_n|$ both convergent. □

Theorem 2.6.7. *If f is continuous on $[-\pi, \pi]$ and has Fourier coefficients $\{a_n\}$, $\{b_n\}$ and $\sum |a_n|$ and $\sum |b_n|$ both converge, then the Fourier series for f, i.e., $\frac{1}{2}a_0 + \sum_1^\infty (a_n cos(nx) + b_n sin(nx))$, converges uniformly to $f(x)$ on $[-\pi, \pi]$.*

Proof. First, the hypothesis ensures that the Fourier series for f converges uniformly on I to some function $g(x)$, necessarily continuous (by the Weierstrass M-test). The relation between the norm $^*\| \ \|$ and the norm $\| \ \|_I$ (see the proof of Theorem 2.3.1) implies that the Fourier series for f converges to g in the metric of $^*C$. By Theorem 2.4.1 the sequences $\{a_n\}$, $\{b_n\}$ must be the Fourier coefficients of g. Accordingly, f and g are both continuous on I and have the same Fourier coefficients. By Theorem 2.6.5 $f - g = 0$, i.e., $f = g$. □

Combine the last two results:

Corollary 2.6.8. *Any periodic function f with continuous derivative on $[-\pi, \pi]$ has a uniformly convergent Fourier series.*

In particular, this applies to the function

$$\frac{x^2}{4} \sim \frac{\pi^2}{12} - \cos(x) + \frac{1}{4}\cos(2x) - \frac{1}{9}\cos(3x) + \cdots$$

With $x = \pi$ we have

$$\begin{aligned} \frac{\pi^2}{4} &= \frac{\pi^2}{12} + 1 + \frac{1}{4} + \frac{1}{9} + \cdots \\ &\Longrightarrow \\ \frac{\pi^2}{6} &= \sum_1^\infty \frac{1}{n^2}. \end{aligned}$$

Corollary 2.6.9. *Any periodic continuous function f on $[-\pi, \pi]$ can be uniformly approximated there by trigonometric polynomials.*

Proof. We know that f can be approximated uniformly by continuous piecewise linear functions, formed from a finite number of line segments. By rounding off the corners, we may uniformly approximate such a piecewise linear function by a function g with a continuous derivative. If f is periodic with $f(\pi) = f(-\pi)$, then g may be chosen to satisfy $g(\pi) = g(-\pi)$ also. And if we now approximate g on $[-\pi, \pi]$ by a partial sum of its uniformly convergent Fourier series, the resulting trigonometric polynomial will be the desired uniform approximation to the function f. □

Corollary 2.6.10. *The trigonometric set $(*)$ is a bsis for ${}^*C$, and every continuous function f has a Fourier series which converges to it "in the mean," that is, in the metric of the space ${}^*C$.*

Proof. We must prove that the trigonometric polynomials are dense in ${}^*C$. By Cor. 2.6.9 they are uniformly dense in the space of periodic continuous functions on $[-\pi, \pi]$, and thus by Theorem 2.3.1 they are also dense in the metric $*\| \ \|$. We need now only observe that the periodic continuous functions are dense in ${}^*C$. For if f is continuous but $f(\pi) \neq f(-\pi)$, then a continuous function g can be chosen which coincides with f exactly, except in a small neighborhood of $x = \pi$, and which is such that $g(-\pi) = g(\pi)$ and ${}^*\|f - g\|$ is as small as desired. □

2.7 Pointwise Convergence of Fourier Series

Until further notice we use the Fourier expansion

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the a_n and b_n are given by Eqs. 2.1, 2.2 and 2.3.

In Example 2.6.1.1 we showed that

$$\sum_{1}^{n} \sin(kx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(n + \frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}.$$

Recall the analog for a sum of cosines using a sequence of functions D_n known as the **Dirichlet kernel**.

For each positive integer n define

$$D_n(t) = \begin{cases} \frac{2n+1}{2\pi} & \text{if } t = 2\pi m,\ m \in Z; \\ \frac{\sin[(n+\frac{1}{2})t]}{2\pi \sin(t/2)} & \text{if } t \neq 2\pi m. \end{cases}$$

Now recall Lemma 1.2.1 as follows.

Lemma 2.7.1.

$$D_n(u) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{n} \cos(ku)\right).$$

The following theorem connects D_n to Fourier series.

Theorem 2.7.2. *Let f be a piecewise continuous and 2π-periodic function. Then*

$$\begin{aligned} S_n f(x) &:= \frac{1}{2}a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right) \\ &= \int_{-\pi}^{\pi} f(x+t) D_n(t) dt. \end{aligned}$$

Proof.

$$\begin{aligned}
S_n f(x) &:= \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_n \cos(nx) + b_n \sin(nx)) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{k=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt\cos(kx) + \\
&+ \sum_{k=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt\sin(kx) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{k=1}^{n}\cos(kt)\cos(kx) + \sin(kt)\sin(kx)\right)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{k=1}^{n}\cos[(k(t-x)]\right)dt,
\end{aligned}$$

where we used the trig identity $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

Now make the substitution $u = t - x$, so $dt = du$, and use the 2π-periodicity of f, to obtain

$$\begin{aligned}
S_n f(x) &= \int_{-\pi-x}^{\pi-x} f(x+u)\frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{n}\cos(ku)\right)du \\
&= \int_{-\pi}^{\pi} f(x+u)D_n(u)du,
\end{aligned}$$

where the last equality follows from Lemma 2.7.1. □

Lemma 2.7.3. *The Dirichlet kernel $\{D_n\}$ satisfies the following properties:*

(i) For each n, D_n is a continuous, 2π-periodic, even function.

(ii) $\int_{-\pi}^{\pi} D_n(t)dt = 1$.

Proof. Part (i) has already been established. For part (ii), let $f(x) = 1$ for all $x \in \mathcal{R}$ and compute the Fourier coefficients using Eqs. 2.1, 2.2 and 2.3. So $a_0 = 2$ and $a_n = b_n = 0$ for $n \geq 1$. By Theorem 2.7.2

$$1 = S_n(f(x) = \int_{-\pi}^{\pi} f(x+t)D_n(t)dt = \int_{-\pi}^{\pi} D_n(t)dt.$$

□

Lemma 2.7.4. *(***The Riemann – Lebesgue Lemma***)*
If f is piecewise continuous on $[a,b]$ and $t \in \mathcal{R}$, then

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx+t)dx = 0.$$

Proof. We may assume that $b-a \leq \pi$. For otherwise we just chop the interval $[a,b]$ into pieces of length at most π and prove the lemma on each piece separately. Translation by a multiple of π does not affect anything except possibly the sign of the integral. So we may assume that $[a,b]$ is contained in $[-\pi,\pi]$. Now extend the definition of f to all of $[-\pi,\pi]$ by setting $f(x)=0$ outside of $[a,b]$. This is still piecewise continuous, and the integrals are unchanged.

Since $f \sim a_0\frac{1}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$, from Parseval's equality (see Remark 2.6.1) we have

$$\begin{aligned}
\frac{1}{\pi}\|f\|^2 &= \frac{1}{\pi}\int_a^b |f(t)|^2 dt \\
&= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right) \\
. &< \infty.
\end{aligned}$$

Since the terms of a convergent series must tend to 0, we have

$$\lim_{n\to\infty} a_n^2 + b_n^2 = 0.$$

So we compute

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} f(x)\sin(nx+t)dx\right| &= \left|\int_a^b f(x)(\cos(nx)\sin(t)+\sin(nx)\cos(t))dx\right| \\
&= \pi\left|a_n\sin(t)+b_n\cos(t)\right| \\
&\leq \pi\sqrt{a_n^2+b_n^2}\sqrt{\sin^2(t)+\cos^2(t)} = \pi\sqrt{a_n^2+b_n^2},.
\end{aligned}$$

which establishes the desired limit. □

If we apply the Riemann – Lebesgue Lemma to the interval $[-\pi,\pi]$ with $t=\pi/2$ and $t=0$, respectively, we find that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =$

0. This is interesting since we really want the Fourier series to converge (pointwise) to the function f, but we have to work harder to discover a sufficient condition for pointwise convergence.

A function f is *piecewise Lipschitz* on the interval I provided f is piecewise continuous on I and there is a constant L such that for each subinterval I_j of continuity, for all x, y in I_j we have $\|f(x)-f(y)\| \leq L\|x-y\|$. A function f is *piecewise* C^1 on $[a,b]$ if f is differentiable except at finitely many points, and f' is piecewise continuous on $[a,b]$.

Lemma 2.7.5. *If f is piecewise C^1 on $[a,b]$, then f is piecewise Lipschitz with constant $L = \|f'\|_\infty$.*

Proof. First note that f' is bounded on each closed interval on which it is continuous. From this it is easy to prove that f' is bounded on $[-\pi,\pi]$. On any interval of continuity for f', the Mean Value Theorem shows that

$$|f(x) - f(y)| \leq \|f'\|_\infty \cdot |x-y|.$$

These estimates can be spliced together when the function is continuous on an interval even if the derivative is not continuous. □

The real goal of this section is the following theorem.

Theorem 2.7.6. The Dirichlet-Jordan Theorem *Suppose that a function f on $\mathcal{R}$ is 2π-periodic and piecewise Lipschitz. If f is continuous at θ, then*

$$\lim_{n\to\infty} S_n f(\theta) = f(\theta).$$

If f has a jump discontinuity at θ, then

$$\lim_{n\to\infty} S_n f(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}.$$

Proof. By part (ii) of Lemma 2.7.3 we have

$$f(\theta^+) = f(\theta^+)\int_{-\pi}^{\pi} D_n(t)dt = 2\int_0^{\pi} f(\theta^+)D_n(t)dt,$$

and a similar equality holds for $f(\theta^-)$:

$$f(\theta^-) = 2\int_0^{\pi} f(\theta^-)D_n(t)dt.$$

By Theorem 2.7.2 we have (using the fact that D_n is an even function):

$$\begin{aligned} S_n f(\theta) &= \int_{-\pi}^{\pi} f(\theta - t) D_n(t) dt \\ &= \int_0^{\pi} (f(\theta + t) + f(\theta - t)) D_n(t) dt. \end{aligned}$$

Combine this with the two previous equalities to obtain

$$S_n f(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} =$$

$$= \int_0^{\pi} \left(f(\theta + t) + f(\theta - t)\right) D_n(t) dt - \int_0^{\pi} \left(f(\theta^+) + f(\theta^-)\right) D_n(t) dt$$

$$= \int_0^{\pi} \left(f(\theta + t) - f(\theta^+)\right) D_n(t) dt + \int_0^{\pi} \left(f(\theta - t) - f(\theta^-)\right) D_n(t) dt.$$

We now consider these two integrals separately. First we prove that

$$\lim_{n \to \infty} \int_0^{\pi} \left(f(\theta + t) - f(\theta^+)\right) D_n(t) dt = 0.$$

Let L be the Lipschitz constant for f, and let $\epsilon > 0$ be given. Choose a positive $\delta < \epsilon / L$ that is so small that f is continuous on $[\theta, \theta + \delta]$. Hence

$$|f(\theta + t) - f(\theta^+)| < Lt \text{ for all } \theta + t \in [\theta, t + \delta].$$

Since $|\sin(t/2)| \geq |t|/\pi$ for all t in $[-\pi, \pi]$, which is an exercise you should work out (see Cor. 1.3.3), it follows that

$$|D_n(t)| = \left| \frac{\sin[(n + \frac{1}{2})t]}{2\pi \sin(t/2)} \right| \leq \frac{1}{2|t|}, \quad -\pi \leq t \leq \pi.$$

Therefore,

$$\left| \int_0^{\delta} \left(f(\theta + t) - f(\theta^+)\right) D_n(t) dt \right| \leq \int_0^{\delta} Lt \frac{1}{2t} dt \leq \frac{L\delta}{2} < \frac{\epsilon}{2}.$$

For the integral from δ to π, we have

$$\int_{\delta}^{\pi} \left(f(\theta + t) - f(\theta^+)\right) D_n(t) dt = \int_{\delta}^{\pi} \frac{f(\theta + t) - f(\theta^+)}{2\pi \sin(t/2)} \sin[(n + \frac{1}{2})t] dt$$

$$= \int_\delta^\pi g(t)\sin[(n+\frac{1}{2})t]dt,$$

where

$$g(t) = \frac{f(\theta+t) - f(\theta^+)}{2\pi\sin(t/2)}.$$

As g is piecewise continuous on $[\delta, \pi]$, we can apply the Riemann-Lebesgue Lemma to this last integral. Thus for all sufficiently large n

$$\left|\int_\delta^\pi \left(f(\theta+t) - f(\theta^+)\right) D_n(t)dt\right| = \left|\int_\delta^\pi g(t)\sin[(n+\frac{1}{2})t]dt\right| < \frac{\epsilon}{2}.$$

Combining these two estimates, we have

$$\left|\int_0^\pi \left(f(\theta+t) - f(\theta^+)\right) D_n(t)dt\right| < \epsilon.$$

Hence the desired limit is zero.

A similar argument shows that the second integral also goes to zero as n goes to infinity. Combining these results proves the theorem.

□

Exercise 2.7.6.1. *Define a function f on $[-\pi, \pi]$ by*

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & \text{if } -\pi \le x \le 0; \\ \frac{1}{2}(\pi - x), & \text{if } 0 < x \le \pi. \end{cases}$$

(i) Show that $f(x) \sim \sum_{n=1}^\infty \frac{1}{n}\sin(nx)$.

(ii) Use the Dirichlet - Jordan Theorem to show that $\sum_{n=1}^\infty \frac{1}{n}\sin(nx)$ converges pointwise to $f(x)$ for $x \in [-\pi, 0) \cup (0, \pi]$, and converges to 0 at $x = 0$. In particular,

$$\frac{\pi - 1}{2} = \sum_{n=1}^\infty \frac{1}{n}\sin(n).$$

Exercise 2.7.6.2. *Define a function f on $[-\pi, \pi]$ by*

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & \text{if } -\pi \le x \le -1; \\ \frac{1}{2}(\pi - 1)x, & \text{if } -1 < x \le 1; \\ \frac{1}{2}(\pi - x), & \text{if } 1 \le x \le \pi. \end{cases}$$

(i) Show that $f(x) \sim \sum_{n=1}^\infty \frac{\sin(n)}{n^2}\sin(nx)$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \sin(nx)$ *converges pointwise to* $f(x)$*. In particular,*

$$\frac{\pi - 1}{2} = \sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n} \right)^2 .$$

(Note: Compare this with the previous exercise.)

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