Solutions to Assignment #4

2.3 #8

Clearly this matrix has 4 pivots, so it must be invertible.

2.3 #18

No, because $Cx = v$ having a solution for every v in $\mathbb{R}^6$, says C must have 6 pivots, thus the map associated with C must be one-to-one. So it is not possible that $Cx = v$ has more than one solution.

2.4 #14

Suppose $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is invertible, then there exists $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ such that $AB = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. So

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

So $A_{22}B_{22} = I$ therefore A_{22} is invertible by theorem 8. Also, $A_{22}B_{21} = 0$ so that $B_{21} = A_{22}^{-1}0 = 0$. So $A_{11}B_{11} + A_{12}B_{21} = I$ and so $A_{11}B_{11} = I$. So A_{11} is invertible by theorem 8.

Now suppose that A_{11} and A_{22} are both invertible. Then

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Therefore A is invertible.

2.5 #12

$$\begin{bmatrix} 2 & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & -14 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}$$

2.5 #26

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P_{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix} P^{-1}$$

$$A^3 = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix} P^{-1}$$

$$A^k = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^k} & 0 \\ 0 & 0 & \frac{1}{3^k} \end{pmatrix} P^{-1}$$

3.1 #10

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)(3)\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$
$$= -3(5(2) + 4(-2))$$
$$= -3(2)$$
$$= -6$$

3.1 #24

$$\begin{vmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{vmatrix} = a(2) - b(6) + c(3)$$
$$= 2a - 6b + 3c$$
$$\begin{vmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{vmatrix} = 3(6b - 5c) - 2(6a - 6c) + 2(5a - 6b)$$
$$= 18b - 15c - 12a + 12c + 10a - 12b$$
$$= -2a + 6b - 3c$$
$$\begin{vmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{vmatrix}$$

The row operation was interchange.

3.2 #12

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ -3 & 0 & -2 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix}$$
$$= 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix}$$
$$= 3\begin{vmatrix} -1 & 2 & 3 \\ 5 & 0 & -3 \\ -3 & 0 & -2 \end{vmatrix}$$
$$= 3(-2)\begin{vmatrix} 5 & -3 \\ -3 & -2 \end{vmatrix}$$
$$= -6(-10 - 9) = -6(-19)$$
$$= 114$$

3.2 #18

$$\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

3.2 #34

$$\det(PAP^{-1}) = \det(P) \cdot \det(A) \cdot \det(P^{-1}) = \det(P) \cdot \det(A) \cdot \frac{1}{\det(P)} = \det(A).$$