

- (1) Using matrices, and elementary row operations, find the general solution to the system of linear equations given below.

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 3 \\2x_1 + 3x_2 + 8x_3 &= 4 \\5x_1 + 8x_2 + 19x_3 &= 11\end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 5 & 8 & 19 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -2 & 4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the general solution is

$$\begin{cases} x_1 = -1 - 7x_3 \\ x_2 = 2 + 2x_3 \\ x_3 \text{ is free} \end{cases}$$

- (2) Determine whether the following vectors are linearly dependent or independent.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has a pivot in every column, therefore  $v_1, v_2,$  and  $v_3$  are linearly independent.

- (3) Consider the map  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by

$$T(x) = Ax \text{ where } A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix}.$$

- a. Determine  $T(x)$  if  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ .

$$T(x) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & 3 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_4 \\ 2x_1 - x_2 + 2x_3 - x_4 \\ x_1 - 3x_2 + 2x_3 - 2x_4 \end{pmatrix}$$

- b. Is this map one-to-one? Why? or Why not?

$A$  has 3 rows, so  $A$  can have at most 3 pivots. Since  $A$  has 4 columns,  $A$  cannot have a pivot in every column. Therefore  $T$  is not one-to-one.

- c. Is this map onto  $\mathbb{R}^3$ ? Why? or Why not?

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & -5 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$A$  does not have a pivot in every row, therefore  $T$  is not onto  $\mathbb{R}^3$ .

- (4) Express the following matrix as a product of elementary matrices:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

This is not the only product that works.

- (5) Give examples of  $3 \times 3$  matrices  $A$  and  $B$  such that  
 a.  $A$  and  $B$  are invertible, but  $A + B$  is **not** invertible.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- b.  $A$  and  $B$  both **not** invertible, but  $A + B$  **is** invertible.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- c.  $A$ ,  $B$ , and  $A + B$  are all invertible.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- (6) Given that  $ad - bc \neq 0$ , find the inverse of the following matrix

$$A = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b & 0 & 0 \\ -c & a & 0 & 0 \\ 0 & 0 & d & -b \\ 0 & 0 & -c & a \end{pmatrix}$$

- (7) Consider the following matrix

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 8 & 9 & 19 \\ 2 & 2 & 5 \end{pmatrix}$$

- a. Compute the  $LU$  factorization of  $A$ .

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 8 & 9 & 19 \\ 2 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

- b. Using the matrix  $L$  found in part a, solve for  $c$  in the matrix equation  $Lc = b$  where  $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- c. Using the matrix  $U$  found in part a, and the vector  $c$  solved for in part b, solve for  $x$  in the matrix equation  $Ux = c$ .

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

- d. Show that the  $x$  that you found in part c, satisfies the matrix equation  $Ax = b$  where  $b$  is defined as in part b.

$$Ax = \begin{pmatrix} 2 & 2 & 4 \\ 8 & 9 & 19 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 6 + 4 \\ 8 - 27 + 19 \\ 2 - 6 + 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b.$$

- e. What part of  $A^{-1}$  is the vector  $x$ ? Explain your answer.

$Ax = b$  has a unique solution, namely  $A^{-1}b$ . Since the  $b$  is the vector  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ , then  $x$  will be the

third column of  $A^{-1}$ .

- (8) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 4 & -3 \\ -1 & 1 & 0 & 2 \\ 3 & 2 & 3 & -1 \\ 1 & -2 & 2 & -3 \end{pmatrix}$$

- a. Find the determinant of the matrix  $A$  given above.

$$\begin{vmatrix} 2 & -1 & 4 & -3 \\ -1 & 1 & 0 & 2 \\ 3 & 2 & 3 & -1 \\ 1 & -2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & 0 & 2 \\ 5 & 0 & 3 & -5 \\ -1 & 0 & 2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 4 & -1 \\ 5 & 3 & -5 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 4 & -1 \\ 0 & -17 & 0 \\ 0 & 6 & 0 \end{vmatrix} = 1 \cdot 1 \cdot \begin{vmatrix} -17 & 0 \\ 6 & 0 \end{vmatrix} = (1)(1)(0) = 0$$

- b. Is the matrix  $A$  invertible? Why? or Why not?

No,  $A$  is not invertible since  $\det(A) = 0$ .

- c. Do the columns of  $A$  span  $\mathbb{R}^4$ ? Why? or Why not?

No, since  $A$  is not invertible, the columns of  $A$  do not span  $\mathbb{R}^4$ .