Math 3000 Intro to Abstract Math
Quiz # 7, Spring 2006

Name: Solutions

1. Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise disjoint families of sets. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$.

   a. Prove that $\mathcal{C}$ is a family of pairwise disjoint sets.

   Proof: Suppose that $C_1, C_2 \in \mathcal{C} = \mathcal{A} \cap \mathcal{B}$. Then $C_1 \in \mathcal{A}$ and $C_2 \in \mathcal{A}$ and since $\mathcal{A}$ is pairwise disjoint, then $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$. Therefore $\mathcal{C}$ is pairwise disjoint. ■

   b. Give an example to show that $\mathcal{D}$ need not be pairwise disjoint.

   Consider the families $\mathcal{A} = \{\{1\}\}$ and $\mathcal{B} = \{\{1, 2\}$, then $\mathcal{D} = \{\{1\}, \{1, 2\}\}$ and is not pairwise disjoint.

   c. Prove that if $\bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{B \in \mathcal{B}} B$ are disjoint, then $\mathcal{D}$ is pairwise disjoint.

   Proof: Let $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$. Then consider the following two cases:

   Case 1: One of $\mathcal{A}$ and $\mathcal{B}$ contains both D_1, and D_2. If this is the case, then since both $\mathcal{A}$ and $\mathcal{B}$ are mutually disjoint, then $D_2 \cap D_2 = \emptyset$.

   Case 2: $\mathcal{A}$ contains one of D_1 and D_2 and $\mathcal{B}$ contains the other. Without loss of generality, we may assume that $D_1 \in \mathcal{A}$ and $D_2 \in \mathcal{B}$. In this case then we have that since $D_1 \subseteq \bigcup_{A \in \mathcal{A}} A$, and $D_2 \subseteq \bigcup_{B \in \mathcal{B}} B$, then

$$D_1 \cap D_2 \subseteq \bigcup_{A \in \mathcal{A}} A \cap \bigcup_{B \in \mathcal{B}} B = \emptyset$$

   since $\bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{B \in \mathcal{B}} B$ are disjoint. Therefore $D_1 \cap D_2 = \emptyset$ in this case as well. ■

2. Use the PMI to prove that for all natural numbers n, $4^n - 1$ is divisible by 3.

   Proof: (Base Case) Consider the case $n = 1$. In this case $4^1 - 1 = 3$ which is divisible by 3.
   (Inductive Step) Now suppose that the result holds for $n = k$ for some integer k. Then $4^k - 1$ is divisible by 3. This means that $4^k - 1 = 3m$ for some integer m. Now consider

$$\begin{aligned} 4^{k+1} - 1 &= 4(4^k) - 1 - 4 + 4 \\ &= 4(4^k - 1) + 3 \\ &= 4(3m) + 3 \\ &= 3(4m + 1) \end{aligned}$$

   where $4m + 1$ is an integer. Therefore 4^{k+1} is divisible by 3 as well. Thus by the PMI, the result holds for all natural numbers n. ■