Math 3000 Intro to Abstract Math
Exam # 2, Spring 2006

Name: **Solutions**

1. Given A, B, and C below (in the universe $\mathbb{R}$), find the specified sets.

   - $A = \{10, 11, 12\}$
   - $B = \{n \in \mathbb{N} \;:\; 5|(n+3)\}$
   - $C = \{C_n : n \in \mathbb{N}\}$, where $C_n = \left(1 - \frac{1}{n}, \frac{n+1}{n}\right]$

   a. $A - B$

   $$\{10, 11\}$$

   b. $\widetilde{A \cap B}$

   $$\begin{aligned} A \cap B &= \{12\} \\ \widetilde{A \cap B} &= \mathbb{R} - \{12\} = \{x \mid x \in \mathbb{R} \wedge x \neq 12\} \end{aligned}$$

   c. $\mathcal{P}(A)$

   $$\mathcal{P}(A)\,\{\emptyset, \{10\}, \{11\}, \{12\}, \{10, 11\}, \{10, 12\}, \{11, 12\}, A\}$$

   d. $\bigcap_{n \in \mathbb{N}} C_n$

   $$\bigcap_{n \in \mathbb{N}} C_n = \{1\}$$

   e. $\bigcap_{n=3}^{10} C_n$

   $$\bigcap_{n=3}^{10} C_n = \left(\frac{9}{10}, \frac{11}{10}\right]$$

2. Use the PMI to show the following:
   For all $n \in \mathbb{N}$, $n^3 + 44n$ is divisible by 3.

   Proof: (Base Case:) Let $n = 1$, then $1^3 + 44 = 45$ which is divisible by 3, so the result holds in this case. (Inductive Step) Now suppose that the result hold for $n = k$ for some natural number k. Then $k^3 + 44k = 3m$ for some integer m. Therefore

   $$\begin{aligned} (k+1)^3 + 44(k+1) &= k^3 + 3k^2 + 3k + 1 + 44k + 44 \\ &= k^3 + 44k + 3(k^2 + k + 15) \\ &= 3m + 3(k^2 + k + 15) \\ &= 3(k^2 + k + 15 + m) \end{aligned}$$

   where $k^2 + k + 15 + m$ is an integer. Thus the result holds for $n = k + 1$, therefore by the PMI the result holds for all natural numbers n. ■

3. Let A and B be sets

   a. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
   Proof: Let $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$, So $x \in A$ and $y \in B$ and $y \in C$. So $(x, y) \in A \times B$ and $(x, y) \in A \times C$, therefore $(x, y) \in (A \times B) \cap (A \times C)$. Thus $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.
   Now let $(x, y) \in (A \times B) \cap (A \times C)$, then $(x, y) \in A \times B$, and $(x, y) \in A \times C$. So $x \in A$ and $y \in B$, and $y \in C$. So $x \in A$ and $y \in B \cap C$. So $(x, y) \in A \times (B \cap C)$. Thus $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Therefore $A \times (B \cap C) = (A \times B) \cap (A \times C)$. ■

b. State the definition of a relation from A to B.
A relation from A to B is a subset of $A \times B$.

c. Give in set builder notation, the definition of the domain of a relation from A to B.
Let R be a relation from A to B

$$\text{Dom }(R) = \{x \in A : \exists y \in B \text{ such that } (x, y) \in R\}$$

d. Give in set builder notation the definition of the range of a relation from A to B.
Let R be a relation from A to B

$$\text{Rng }(R) = \{y \in B : \exists x \in A \text{ such that } (x, y) \in R\}$$

4. Let $A = \{1, 2, 3, 4\}, B = \{3, 7, 8, 9\}$, and $C = \{a, b, c, d\}$. Also, let $R = \{(1, 9), (1, 3), (3, 3), (4, 8)\}$ and $S = \{(3, b), (7, c), (8, a), (9, d)\}$. Find

a. Dom(R)
$$= \{1, 3, 4\}$$

b. Rng(R)
$$= \{3, 8, 9\}$$

c. Dom(S)
$$= \{3, 7, 8, 9\}$$

d. Rng(S)
$$= \{a, b, c, d\}$$

e. $S \circ R$
$$= \{(1, d), (1, b), (3, b), (4, a)\}$$

f. Dom($S \circ R$)
$$= \{1, 3, 4\}$$

g. Rng($S \circ R$)
$$= \{a, b, d\}$$

h. S^{-1}
$$= \{(b, 3), (c, 7), (a, 8), (d, 9)\}$$

i. Dom(S^{-1})
$$= \{a, b, c, d\}$$

j. Rng(S^{-1})
$$= \{3, 7, 8, 9\}$$

5. For each of the following relations R_i on $\{a, b, c\}$, circle if it is reflexive, symmetric, and/or transitive.

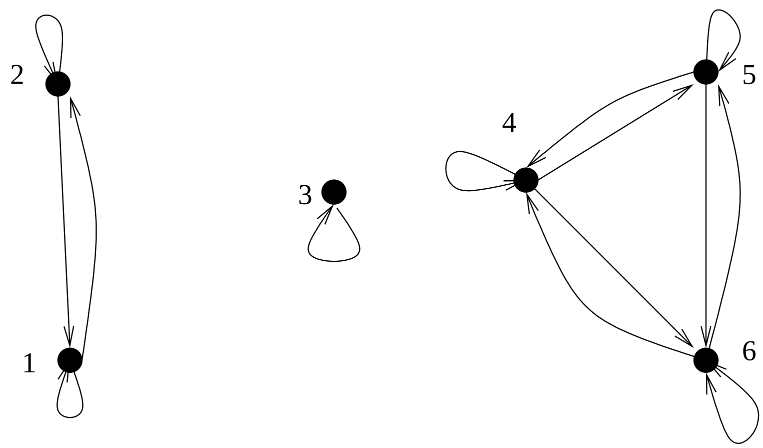
$R_1 = \{(a, b), (b, a)\}$	reflexive	**symmetric**	transitive
$R_2 = \{(a, a), (a, b), (b, b), (b, c), (a, c)\}$	reflexive	symmetric	**transitive**
$R_3 = \{(c, c)\}$	reflexive	**symmetric**	**transitive**

6. Let $A = \{1, 2, 3, 4, 5, 6\}$

a. Which of the following is a partition of A? For those that are not, why not?

- $\{\{1, 2\}, \{3\}, \{4, 5\}\}$
  Not a partition. 6 is not in any element of the set.
- $\{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$
  Is a partition.
- The power set of A
  Not a partition, because it contains the empty set.

b. Draw the digraph for the relation R associated to the partition in part a.



c. What is $6/R$?

$$6/R = \{4, 5, 6\}$$

7. Let $A = \{1, 2, 3\}$, and consider the relation on $\mathcal{P}(A)$ given by XRY iff $X \subseteq Y$, where X and Y are elements of $\mathcal{P}(A)$.

a. Show that R is a partial ordering for $\mathcal{P}(A)$.
Proof: Let $X \in \mathcal{P}(A)$, then $X \subseteq X$ by definition. Therefore XRX. So R is reflexive.
Now let $X, Y \in \mathcal{P}(A)$ with XRY and YRX, then $X \subseteq Y$ and $Y \subseteq X$, so $X = Y$. Therefore R is anti-symmetric.
Finally let $X, Y, Z \in \mathcal{P}(A)$ such that XRY, and YRZ. Then $X \subseteq Y$, and $Y \subseteq Z$, therefore $X \subseteq Z$, so XRZ. Thus R is transitive. Therefore R is a partial ordering for $\mathcal{P}(A)$. ■

b. List the immediate predecessors of $\{1, 2\}$.
The immediate predecessors of $\{1, 2\}$ are $\{1\}$ and $\{2\}$.

c. What is the supremum of $\mathcal{P}(A)$ under this ordering? (Explain your answer.)
Sup $\mathcal{P}(A) = A$. For any $X \in \mathcal{P}(A), X \subseteq A$. So A is an upper bound for $\mathcal{P}(A)$. Also for any upper bound Y of $\mathcal{P}(A)$, $A \subseteq Y$ since $A \in \mathcal{P}(A)$. So $A = $ Sup $\mathcal{P}(A)$.

d. What is the infimum of $\mathcal{P}(A)$ under this ordering? (Explain your answer.)
Inf $\mathcal{P}(A) = \emptyset$. For any $X \in \mathcal{P}(A), \emptyset \subseteq X$, so $\emptyset$ is a lower bound for $\mathcal{P}(A)$. Also since $\emptyset \in \mathcal{P}(A)$, then for any lower bound Y of $\mathcal{P}(A)$ we must have $Y \subseteq \emptyset$. So Inf $\mathcal{P}(A) = \emptyset$.

8. Let $F : \mathbb{Z} \to \mathbb{Z}_{11}$ be the map defined by $F(z) = \bar{z}$ where $\bar{z}$ represents the equivalence class of z under the relation $\equiv_{11}$. (The integers mod 11). In other words $F(z) = z/\equiv_{11}$. Circle the integers from the list which have the same image as 7 under F.

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