Infinite Sets

Recall:

1. A set S is *finite* iff $S = \emptyset$, or $S \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. A set S is *infinite* if it is not finite.

2. If a set A is finite, then A is not equivalent to any of its proper subsets.

3. If a set A is equivalent to any of its proper subsets, then A is infinite.

Definition: A set S is *denumerable* if $S \approx \mathbb{N}$. We write $\overline{\overline{S}} = \aleph_0$.

Theorem: The set of integer $\mathbb{Z}$ is denumerable.

proof: Consider the function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \text{ is even} \\ \frac{1}{2}(1-x), & \text{if } x \text{ is odd} \end{cases}$$

(Show f is 1-1): Assume $f(x) = f(z)$ for some $x, z \in \mathbb{N}$.

Case 1: Suppose x and z are both even, then

$$\begin{aligned} \frac{1}{2}x &= \frac{1}{2}z \quad \Rightarrow \\ x &= z \end{aligned}$$

Case 2: Assume x and z are both odd. Then

$$\begin{aligned} \frac{1}{2}(1-x) &= \frac{1}{2}(1-z) \quad \Rightarrow \\ 1-x &= 1-z \quad \Rightarrow \\ -x &= -z \quad \Rightarrow \\ x &= z \end{aligned}$$

Case 3:Assume x is even and z is odd. Then if this is the case, then $f(x) = \frac{1}{2}x \geq 0$, while

$f(z) = \frac{1}{2}(1 - z) < 0$, therefore this case cannot happen.
Therefore in all possible cases, $f(x) = f(z) \Rightarrow x = z$. So f is 1-1.

(Show f is onto $\mathbb{Z}$.) Let $w \in \mathbb{Z}$.
Case 1: Assume $w > 0$, then let $x = 2w$. Then x is even, and $f(x) = \frac{1}{2}x = \frac{1}{2}(2w) = w$.
Case 2: Assume $w \leq 0$, then let $x = 1 - 2w$. Then x is odd, and $f(x) = \frac{1}{2}(1 - x) = \frac{1}{2}(1 - (1 - 2w)) = \frac{1}{2}(2w) = w$.
Therefore f is onto $\mathbb{Z}$.
Thus $\mathbb{Z} \approx \mathbb{N}$ and therefore $\mathbb{Z}$ is denumerable.

$\square$

Theorem: The set $\mathbb{N} \times \mathbb{N} = \{(m, n) \ : \ m \in \mathbb{N} \ \text{ and } \ n \in \mathbb{N}\}$ is denumerable.
proof: Consider the function

$$f(m, n) = 2^{m-1}(2n - 1).$$

Definition: A set S is *countable* iff it is finite or denumerable. Otherwise S is *uncountable.*

Theorem: The open interval $(0, 1)$ is uncountable.