Equivalent Sets

In this section we address the issue of the "size" of a set.

Definition: Two sets A and B are *equivalent* iff there exists a 1-1 function from A onto B.

Note:

1. We say A and B are in 1-1 correspondence.

2. We write $A \approx B$.

Example: $\{1, 2, 3\} \approx \{a, b, g\}$ since the function $f : \{1, 2, 3\} \to \{a, b, g\}$ given by $f(1) = a$, $f(2) = g$, and $f(3) = b$ is 1-1 and onto $\{a, b, g\}$.

Example: Are $\{1, 2, 3, 4, \ldots\}$ and $\{3, 6, 9, 12, \ldots\}$ equivalent?

Note: $\{3, 6, 9, 12, \ldots\} \subset \{1, 2, 3, 4, \ldots\}$, but yet they are still equivalent.

Example: Are the intervals $[1, 2]$ and $[1, 4]$ equivalent?

Worksheet Example A

The following theorem will be useful:

Theorem: If $f : A \underset{\text{onto}}{\overset{\text{1-1}}{\longrightarrow}} B$ and $g : B \underset{\text{onto}}{\overset{\text{1-1}}{\longrightarrow}} C$, then $g \circ f : A \underset{\text{onto}}{\overset{\text{1-1}}{\longrightarrow}} C$.

Theorem: The relation "$\approx$" is an equivalence relation on the set of all sets.

Theorem: Assume A, B, C, and D are sets with $A \approx C$ and $B \approx D$. Then

1. $A \times B \approx C \times D$.

2. If $A \cap B = \emptyset$ and $C \cap D = \emptyset$, then $A \cup B \approx C \cup D$.

Example: Let $A = \{-5, 0\}$, $B = [1, 2]$, $C = \{9, 101\}$, and $D = [1, 4]$. Then $A \approx C$ and $B \approx C$. By the previous theorem

$$\{-5, 0\} \cup [1, 2] \approx \{9, 101\} \cup [1, 4]$$

Notation: Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and let $\mathbb{N}_k = \{1, 2, 3, \dots, k\}$ for $k \in \mathbb{N}$.

Definition: A set S if *finite* iff $S = \emptyset$ or $S \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. Set S is *infinite* if it is not finite.

Note:

1. We say the empty set $\emptyset$ has *cardinality zero*.

2. If $S \approx \mathbb{N}_k$ we say set S has *cardinality* k.

Notation: $\overline{\overline{S}}$ means "cardinality of set S."

Worksheet Example B

Lemma: Assume set S is finite with cardinality k, and $x \notin S$. then $S \cup \{x\}$ is finite with cardinality $k+1$.

Lemma: Every subset of $\mathbb{N}_k$ is finite for all $k \in \mathbb{N}$.

Theorem: Assume S is a finite set. If $T \subseteq S$, then T is finite. In other words, every subset of a finite set is finite.

The proof of our last theorem gives us the following corollary:

Corollary: Any set, which is equivalent to a finite set, must also by finite.

The Contrapositive of the previous theorem says:

If a subset of set a A is infinite, then A must be infinite.

Theorem: If sets A and B are finite with $A \cap B = \emptyset$, then $A \cup B$ is finite and $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}}$.

Corollary:

a. If A and B are finite sets, then $A \cup B$ is finite.

b. If $A_1, A_2, A_3, \ldots, A_n$ are finite sets for $n \in \mathbb{N}$, then $\bigcup_{i=1}^{n} A_i$ is finite.

We say "the union of any finite collection of finite sets is finite."

Lemma: If $r > 1$ and $x \in \mathbb{N}_r$, then $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$.

Theorem: (The Pigeonhole Principle) Let $n, r \in \mathbb{N}$. If $f : \mathbb{N}_n \to \mathbb{N}_r$ is a function, and $n > r$, then f is not 1-1.

Corollary: If a set is finite, then it is not equivalent to any of its proper subsets.

This gives us the statement: If a set S is equivalent to any of its proper subsets, then S is infinite.

Example: We claim that $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is infinite. To show this, we can show that $\mathbb{N}$ is equivalent to one of its proper subsets. Consider the set $A = \{2, 3, 4, \ldots\}$. It is clear that $A \subset \mathbb{N}$. However consider the function $f : \mathbb{N} \to A$ given by $f(n) = n+1$ for all $n \in \mathbb{N}$. This function is clearly 1-1 and onto A. Therefore $\mathbb{N} \approx A$ and thus $\mathbb{N}$ is infinite.

Worksheet Example C