Surjective and Injective Functions

Recall that a function $f : A \to B$ satisfies

i. $\text{Dom}(f) = A$, and

ii. $\text{Rng}(f) \subseteq B$.

Definition: A function $f : A \to B$ is *onto* B iff $\text{Rng}(f) = B$.

Worksheet Example A

Remarks: Assume $f : A \to B$ is onto B.

1. Write $f : A \overset{\text{onto}}{\to} B$.

2. Call f a *surjection.*

3. Alternate definition: For all $y \in B$, there exists an $x \in A$ so that $f(x) = y$.

4. In general $f : A \to B$ satisfies $\text{Rng}(f) \subseteq B$, but $f : A \overset{\text{onto}}{\to} \text{Rng}(B)$ is always true.

How do we characterize a function $f : A \to B$ which is NOT onto B?

$$
\begin{aligned}
& \sim [(\forall y \in B)(\exists x \in A)(\text{so that } f(x) = y)] \\
\iff \quad & (\exists y \in B) \sim (\exists x \in A)(\text{so that } f(x) = y) \\
\iff \quad & (\exists y \in B)(\forall x \in A) \sim (\text{so that } f(x) = y) \\
\iff \quad & (\exists y \in B)(\forall x \in A)(f(x) \neq y)
\end{aligned}
$$

In other words there is at least one $y \in B$ without a pre-image $x \in A$.

Example: Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 3x^5 - 7$ is onto $\mathbb{R}$.
proof:

Let $y \in \mathbb{R}$. Then consider $x = \left[\frac{1}{3}(y+7)\right]^{\frac{1}{5}}$.
Then

$$\begin{aligned} f(x) &= f(\left[\frac{1}{3}(y+7)\right]^{\frac{1}{5}}) \\ &= 3 \cdot \left[\frac{1}{3}(y+7)\right] - 7 \\ &= (y+7) - 7 \\ &= y \end{aligned}$$

Thus f is onto $\mathbb{R}$.

$\square$

Example: Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5(\sin x)^3 + 8$ is not onto $\mathbb{R}$.
proof:
Let $y = 14$. Then

$$\begin{aligned} f(x) = 14 &\iff 5(\sin x)^3 + 8 = 14 \\ &\iff 5(\sin x)^3 = 6 \\ &\iff (\sin x)^3 = \frac{6}{5} \\ &\iff \sin x = \left(\frac{6}{5}\right)^{\frac{1}{3}} > 1 \end{aligned}$$

Of course this cannot happen, therefore $y = 14$ has no pre-image $x \in \mathbb{R}$. Thus f is not onto $\mathbb{R}$.

$\square$

Worksheet Example B

Theorem: If functions $f : A \stackrel{\text{onto}}{\rightarrow} B$ and $g : B \stackrel{\text{onto}}{\rightarrow} C$, then function $g \circ f : A \stackrel{\text{onto}}{\rightarrow} C$.

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Worksheet Example C

Definition: A function $f : A \rightarrow B$ is *one-to-one* iff $f(x) = f(z) \;\Rightarrow\; x = z$.

Remarks: Assume $f : A \to B$ is one-to-one.

1. Write $f : A \overset{1\text{-}1}{\to} B$.

2. Call f an *injection.*

3. If pairs in f can be graphed in the Cartesian plane, then f passes the "horizontal line test."

4. Alternate definition: f is 1-1 iff $x \neq z \;\Rightarrow\; f(x) \neq f(z)$.

How do we characterize a function $f : A \to B$ which is NOT 1-1?
Find $x, z \in A$ with $f(x) = f(z)$, but $x \neq z$.

Worksheet Example D

Recall: Let the function $f : A \to B$ be given. The inverse relation f^{-1} from B to A is

$$f^{-1} = \{(x, y) \in B \times A \ : \ (y, x) \in f\}.$$

Theorem: Let the function $f : A \to B$ be given. Then the inverse relation f^{-1} from Rng(f) to A is a function iff f is 1-1. Furthermore, if f^{-1} is a function, then f^{-1} is 1-1.

Theorem: If functions $f : A \overset{\text{1-1}}{\to} B$ and $g : \overset{\text{1-1}}{\to} C$, then $g \circ f : A \overset{\text{1-1}}{\to} C$.

Theorem: If functions $f : A \to B$, $g : B \to C$, and $g \circ f : A \overset{\text{1-1}}{\to} C$, then $f : A \overset{\text{1-1}}{\to} B$.

Definition: A function $f : \underset{\text{onto}}{\overset{\text{1-1}}{\to}} B$ is called a *bijection* or *one-to-one correspondence.*

Worksheet Example E