Combinations of Functions

We begin by recalling a couple of definitions.

Definition: Let A and B be sets. A *relation* R from A to B is a subset of $A \times B$.

Definition: Let A and B be sets. A *function* f from A to B is a relation f from A to B ($f \subseteq A \times B$) which satisfies the following two properties:

i. Dom$(f) = A$.

ii. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Definition: Assume $f : A \to B$ is a function, i.e.,

$$f = \{(x, y) \in A \times B \ : \ f(x) = y\}.$$

Then the *inverse* **relation** f^{-1} is

$$f^{-1} = \{(x, y) \ : \ (y, x) \in f\}.$$

Worksheet Example A

Remarks: Assume f is a function from A to B, and f^{-1} is the inverse relation of f.

1. f^{-1} need NOT be a function.

2. $f \subseteq A \times B$ and $\text{Dom}(f) = A$.

3. $f^{-1} \subseteq B \times A$ and $\text{Dom}(f^{-1}) = \text{Rng}(f)$.

4. $\text{Rng}(f^{-1}) = \text{Dom}(f) = A$.

Definition: Assume $f : A \to B$, and $g : B \to C$. The *composition of f and g* is the relation

$$
\begin{aligned}
g \circ f \;=\; & \{(x,z) \in A \times C \;:\; \exists y \in B \text{ with } (x,y) \in f \\
& \text{and } (y,z) \in g\}.
\end{aligned}
$$

Worksheet Example B

Note:

$$
\begin{aligned}
g \circ f \;=\; & \{(x,z) \in A \times C \;:\; \exists y \in B \text{ with } (x,y) \in f \\
& \text{and } (y,z) \in g\}. \\
=\; & \{(x,z) \in A \times C \;:\; \exists y \in B \text{ with } f(x) = y \\
& \text{and } g(y) = z\} \\
=\; & \{(x,z) \in A \times C \;:\; g(f(x)) = z\}.
\end{aligned}
$$

Worksheet Example C

Remark: In general, $g \circ f \neq f \circ g$.

Note: In all examples in this section $f \circ g$ and $g \circ f$ are functions.

Is this always true?

Theorem: Assume $f : A \to B$ and $g : B \to C$ are functions. Then the relation $g \circ f$ from A to C satisfies:

1. $\text{Dom}(g \circ f) = \text{Dom}(f) = A$. and

2. $g \circ f$ is a function, i.e., $g \circ f : A \to C$.

We can now say $(x, y) \in g \circ f$ iff $(g \circ f)(x) = y$ and $(g \circ f)(x) = g(f(x))$.

Note: If $f : A \to B$ and $g : B \to C$ are functions, it follows that $\text{Rng}(f) \subseteq \text{Dom}(g)$.

Recall: Two functions f and g are equal, i.e. $f = g$ iff

i. $\text{Dom}(f) = \text{Dom}(g)$, and

ii. $f(x) = g(x)$ for all $x \in \text{Dom}(f)$.

Theorem: Let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are functions and

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Remark: It has now been established that functional composition is associative, but not commutative.

Definition: (Identity function on A) Let A be a nonempty set. The *identity function on* A, $I_A : A \to A$, is defined by

$$I_A(x) = x$$

for all $x \in A$.

Theorem: Let $f : A \to B$ be a function and let I_A and I_B be the identity functions on A and B respectively. Then

1. $f \circ I_A = f$, and

2. $I_B \circ f = f$.

Theorem: Let $f : A \to B$ be a function with $\text{Rng}(f) = C$. If f^{-1} is a function, then

1. $f^{-1} \circ f = I_A$, and

2. $f \circ f^{-1} = I_C$.

Definition: Let $f : A \to B$ be a function and let $D \subseteq A$. The *restriction of f to D*, denoted $f|_D$, is given by

$$f|_D = \{(x, y) \in A \times B \ : \ x \in D \text{ and } f(x) = y\}.$$

If f and g are functions, and g is a restriction of f, then we say f is an *extension* of g.

Worksheet Example D

Theorem: Assume h is a function with $\text{Dom}(h) = A$ and g is a function with $\text{Dom}(g) = B$. If

$A \cap B = \emptyset$, then $h \cup g$ is a function with Dom$(h \cup g) = A \cup B$.

Worksheet Example E