Functions

We begin by recalling a couple of definitions.

Definition: Let A and B be sets. A *relation* R from A to B is a subset of $A \times B$.

Definition: Let A and B be sets. Assume R is a relation from A to B, i.e., $R \subseteq A \times B$.

1. The *domain* of R is

   $\text{Dom}(R) = \{x \in A \ : \ \exists y \in B \text{ with } (x, y) \in R\}$.

2. The *range* of R is

   $\text{Rng}(R) = \{y \in B \ : \ \exists x \in A \text{ with } (x, y) \in R\}$.

We now add a couple of properties to relations to make them functions.

Definition: Let A and B be sets. A *function* f from A to B is a relation f from A to B ($f \subseteq A \times B$) which satisfies the following two properties:

i. $\text{Dom}(f) = A$.

ii. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Worksheet Example A

Notation: If f is a function from A to B, then we write "$f : A \to B$" and use the notation

$$(x, y) \in f \quad \Longleftrightarrow \quad x \ f \ y \quad \Longleftrightarrow f(x) = y.$$

Remarks: Assume f is a function from A to B.

1. The word "function" is interchangeable with the word "mapping."

2. Set A is the *domain* of f and set B is called the *codomain* of f.

3. The *range* of f is the set

   $\text{Rng}(f) = \{y \in B \ : \ \exists x \in A \text{ with } (x, y) \in f\}$

4. Each $x \in A$ is paired with **exactly one** element of B.

5. If the ordered pairs in f are points on a graph in the Cartesian plane (e.g., $f \subseteq \mathbb{R} \times$

$\mathbb{R}$), then the graph "passes the vertical line test."

Definition: Assume $f : A \to B$. If $(x, y) \in f$, then we write $f(x) = y$ and

i. call y the *value* of f at x.

ii. call y this *image* of x under f.

iii. call x the *pre-image* of y under f.

Note: Two functions $f : A \to B$ and $g : A \to B$ are equal, written $f = g$, iff

i. $f \subseteq g$, and

ii. $g \subseteq f$.

The following theorem gives us a more useful way to prove that functions are equal.

Theorem: Let $f : A \to B$ and $g : C \to D$ be functions. These functions are equal, i.e., $f = g$ iff

i. Dom$(f) =$ Dom(g), i.e., $A = C$, and

ii. $f(x) = g(x)$ for all $x \in A$.

proof: ($\Rightarrow$) Assume $f = g$. Let us show that $A = C$, and that $f(x) = g(x)$ for all $x \in A$.
To show $A = C$, we let $x \in A = \text{Dom}(f)$. Then there exists some $y \in B$ such that $(x, y) \in f$. However, since $f = g$, then $(x, y) \in g$. Therefore $x \in \text{Dom}(g) = C$. So $A \subseteq C$.
Now let $x \in C = \text{Dom}(g)$, then there exists a $y \in D$. such that $(x, y) \in g$. Thus $(x, y) \in f$ since $g = f$. So $x \in \text{Dom}(f) = A$. Therefore $C \subseteq A$, so $A = C$.
Now we show that $f(x) = g(x)$ for all $x \in A$. To do this, pick $x \in A$ arbitrarily. Then $x \in \text{Dom}(f)$ so there exists some $y \in B$ such that $(x, y) \in f$. Similarly since $\text{Dom}(g) = C = A$, then there exists some $z \in D$ such that $(x, z) \in g$. But since $f = g$, then $(x, z) \in f$ as well. However since f is a function, then $y = z$. So what we have shown is that $f(x) = y = z = g(x)$ as desired.
($\Leftarrow$) Please do this direction for homework.

$\square$

Following are definitions of particular functions with particular notation. **Definition:** (Characteristic function) Let U be a universe and let $A \subseteq U$. Define

$$\psi_A : U \to \{0, 1\}$$

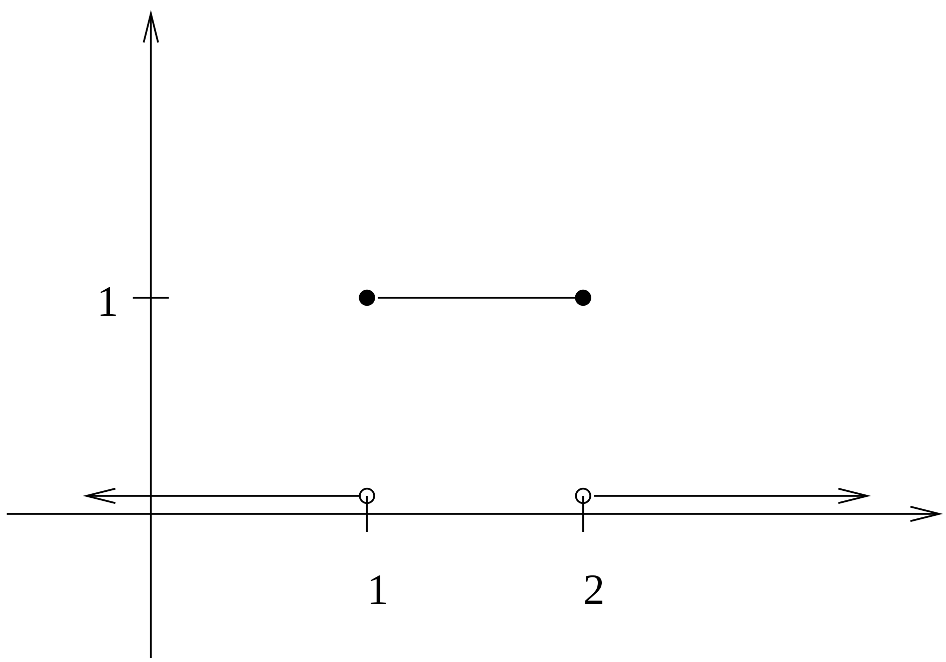
by

$$\psi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in U - A = \widetilde{A}. \end{cases}$$

Example:

1. Let $U = \mathbb{R}$ and let $A = [1, 2] \subseteq \mathbb{R}$. Then

$$\psi_A(x) = \begin{cases} 1 & \text{if } x \in [1, 2] \\ 0 & \text{if } x \in \mathbb{R} - [1, 2]. \end{cases}$$


1
1
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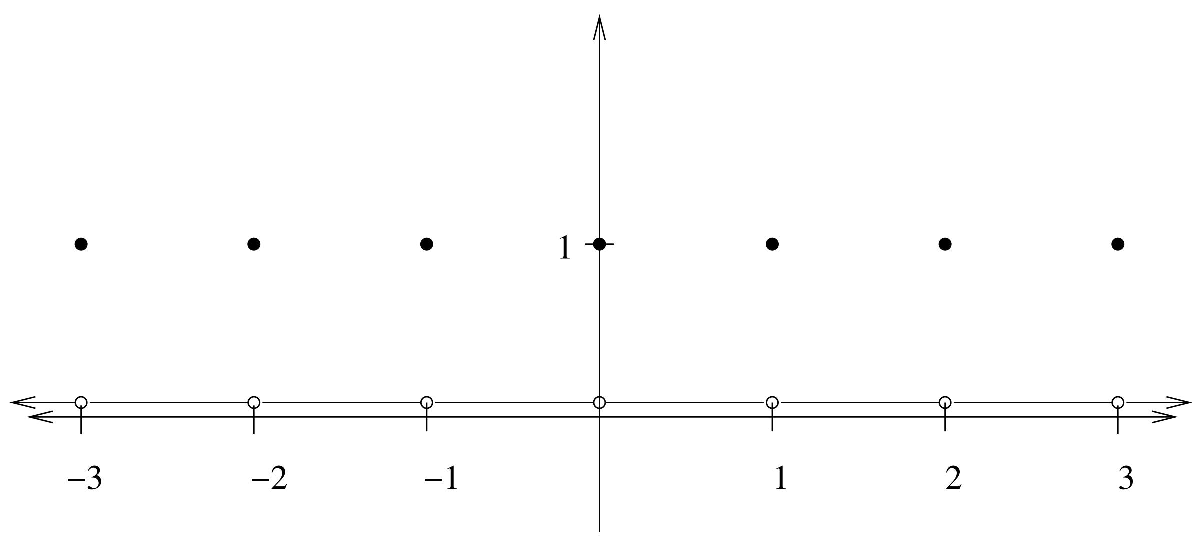
2. Let $U = \mathbb{R}$ and let $A = \mathbb{Z} \subseteq \mathbb{R}$. Then

$$\psi_A(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Z}. \end{cases}$$



Definition: (Projection functions) Let S be a relation from A to B (i.e. $S \subseteq A \times B$. Define $\pi_1 : S \to A$ by

$$\pi_1(a, b) = a,$$

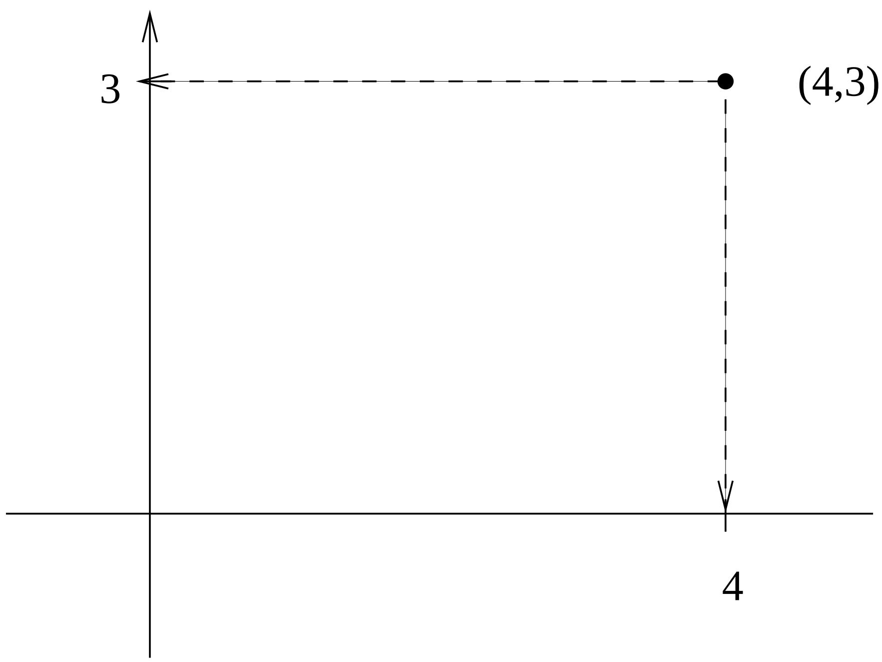
and define $\pi_2 : S \to B$ by

$$\pi_2(a, b) = b$$

for all $(a, b) \in S$.

Example: Let S be a relation on $\mathbb{R}$ and assume $(4, 3) \in S$. Then

$$\pi_1(4,3) = 4 \quad \text{and} \quad \pi_2(4,3) = 3.$$



Worksheet Example B