Partially Ordered Relations

Definition: A relation R on a set A is *antisymmetric* if, for all $x, y \in A$, if $x\ R\ y$ and $y\ R\ x$, then $x = y$.

This gives us a new type of relation.

Definition: A relation R on a set A is a *partial order* for A if R is

a. reflexive on A: $x\ R\ x$ for all $x \in A$.

b. antisymmetric: If $x\ R\ y$ and $y\ R\ x$, then $x = y$.

c. transitive: If $x\ R\ y$, and $y\ R\ z$, then $x\ R\ z$.

The set A is called a *partially ordered set* or *poset*.

We are more familiar with these than you think.

Consider the relation "$\leq$" on $\mathbb{R}$. This is a partial order, and thus $\mathbb{R}$ is a poset.

Example: Consider the relation "|" "divides" on $\mathbb{N}$. Let us show that "divides" is a partial order on $\mathbb{N}$.
proof: (Reflexive) Let $a \in \mathbb{N}$, then $a = 1 \cdot a$ so $a \mid a$.
(Antisymmetric) Now suppose $a, b \in \mathbb{N}$ with $a \mid b$, and $b \mid a$. Then $b = ak$ and $a = bn$ for integers n and k. Substituting gives us that $b = bnk$. So $nk = 1$, therefore either $n = k = 1$, or $n = k = -1$. However if $n = k = -1$, then $a = -b$ and since a and b are both natural numbers, then this is impossible. Therefore $n = k = 1$ and thus $a = b$ as desired.

(Transitive:) Now suppose that $a, b, c \in \mathbb{N}$ with $a \mid b$ and $b \mid c$. Then $b = ak$, and $c = bn$ for some integers n and k. Again substitution gives us that $c = akn$ and since kn is an integer then $a \mid c$.

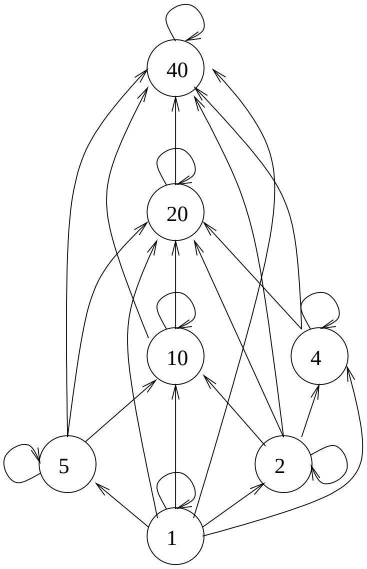
$\square$

Worksheet Example A

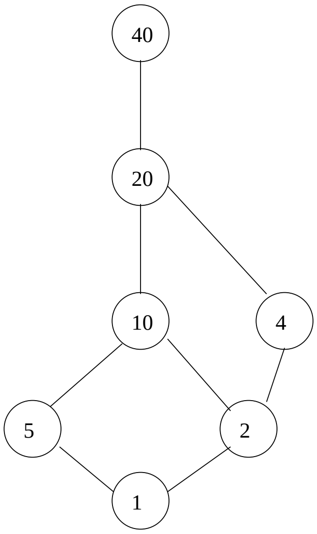
Definition: Let R be a partial ordering on a set A and let $a, b \in A$ with $a \neq b$. Then a is an *immediate predecessor* of b if $a \; R \; b$, and there **does not** exist $c \in A$ such that $a \neq c$, $b \neq c$, $a \; R \; c$, and $c \; R \; b$.
Another way to put this: If a is an immediate predecessor of b and $c \in A$ with $a \; R \; c$, and $c \; R \; b$, then $c = a$, or $c = b$.

Example: Consider the divisors of 40.
The immediate predecessors for 20 are 4 and 10.
The immediate predecessors for 10 are 5 and 2.
The immediate predecessor for 5 is 1.
The immediate predecessor for 1 is 1.

Hasse Diagrams from Relation Digraphs
Let us draw the digraph for the preceding relation (divides) on the set of all divisors of 40 ($\{1, 2, 4, 5, 10, 20, 40\}$).



This becomes



The arrows and extra edges can be omitted, because they are implied by the properties of a partial order.

Hasse Diagrams can only be used for partial orders.

Worksheet Example B

More Definitions

Definition: Let R be a partial order for A and let B be any subset of A. Then $a \in A$ is an *upper bound* for B if for every $b \in B$, $b\ R\ a$. Also a is called a *least upper bound (or supremum)* for B if

1. a is an upper bound for B, and

2. $a\ R\ x$ for every upper bound x for B.

Definition: Let R be a partial order for A and let B be any subset of A. Then $a \in A$ is a *lower bound* for B if for every $b \in B$, $a\ R\ b$. Also a is called a *greatest lower bound (or infimum)* for B if

1. a is a lower bound for B, and

2. $x \; R \; a$ for every lower bound x for B.

If a greatest lower bound or least upper bound for a set exists, and they lie in the set, then we call them the smallest and largest elements of the set respectively.

Definition: A partial ordering R on A is called a *linear order* (or total order) on A if for any two elements $x, y \in A$, either $x \; R \; y$ or $y \; R \; x$.

Example: The partial order "$|$" on $\mathbb{N}$ is **not** a linear order.
The partial order "$\leq$" on $\mathbb{R}$ is a linear order.

Definition: Let L be a linear ordering on a set A. L is a *Well ordering* on A if every nonempty subset B of A contains a smallest element.