# Partitions

**Recall:** Relation $R$ on set $A$ ($R \subseteq A \times A$) is an *equivalence relation on* $A$ iff $R$ is

a. reflexive on $A$: $x \; R \; x$ for all $x \in A$.

b. symmetric: If $x \; R \; y$, then $y \; R \; x$.

c. transitive: If $x \; R \; y$, and $y \; R \; z$, then $x \; R \; z$.

**Recall:** If $R$ is an equivalence relation on $A$, then for $x \in A$, $x/R = \{y \in A \ : \ x \; R \; y\}$ is the equivalence class of $x$. (Read "$x$ mod $R$.")

**Example:** If $A = \{3, 4, 5\}$, then $R = \{(3, 3), (4, 4), (5, 5), (3, 5), (5, 3)\}$ is an equivalence relation on $A$ with equivalence classes

$$\begin{aligned} 3/R &= \{3, 5\} \\ 4/R &= \{4\} \\ 5/R &= \{3, 5\}. \end{aligned}$$

# Worksheet Example A

**Theorem:** Let $R$ be an equivalence relation on a nonempty set $A$. Then

a. $x \in x/R$ for all $x \in A$.

b. $x/R \subseteq A$ for all $x \in A$.

c. $\bigcup_{x \in A} x/R = A$.

d. $x\ R\ y$ iff $x/R = y/R$.

e. $x\ \not R\ y$ iff $x/R \cap y/R = \emptyset$.

proof: Assume that $R$ is an equivalence relation on set $A$.

a. Let $x \in A$. Since $R$ is reflexive on $A$, then $x\ R\ x$. Therefore $x \in x/R$ by definition.

b. Let $x \in A$, and let $z \in x/R$, then $z \in \{w \in A\ :\ x\ R\ w\}$. Thus $z \in A$. Therefore $x/R \subseteq A$.

c. ($\subseteq$) Let $z \in \bigcup\limits_{x \in A} x/R$, then $z \in x/R$ for some $x \in A$. Since $x/R \subseteq A$, then $z \in A$ as well. Thus $\bigcup\limits_{x \in A} x/R \subseteq A$.

($\supseteq$) Let $z \in A$, then $z \in z/R$, therefore $z \in \bigcup_{x \in A} x/R$. Therefore $A \subseteq \bigcup_{x \in A} x/R$.

d. ($\Rightarrow$) Assume that $x \ R \ y$, now we show that $x/R = y/R$.
($\subseteq$) Let $z \in x/R$, then $x \ R \ z$. However since $R$ is symmetric, then $z \ R \ x$. We also have that $x \ R \ y$, so by transitivity $z \ R \ y$, and using symmetry again gives us $y \ R \ z$, therefore $z \in y/R$. Thus $x/R \subseteq y/R$.
($\supseteq$) Now let $z \in y/R$, then $y \ R \ z$. But since $x \ R \ y$, and $R$ is transitive, then $x \ R \ z$ and so $z \in x/R$. Therefore $y/R \subseteq x/R$ as well.
($\Leftarrow$) Now assume that $x/R = y/R$, and we will show that $x \ R \ y$.
Since $y \in y/R$, and $y/R = x/R$, then $y \in x/R$. This gives us that $x \ R \ y$ as desired.

e. ($\Rightarrow$) We first assume that $x \not R \ y$ and we show that $x/R \cap y/R = \emptyset$. We do this by contrapositive. Suppose that $z \in x/R \cap y/R$, then $z \in x/R$, and $z \in y/R$. So $x\ R\ z$ and $y\ R\ z$. But since $R$ is symmetric, then $z\ R\ y$ as well. However since $R$ is transitive, then $x\ R\ y$. So we have shown the desired result.

($\Leftarrow$) Now we assume that $x/R \cap y/R = \emptyset$, and show that $x \not R\ y$. Again we do this by contrapositive. Suppose that $x\ R\ y$, then $y \in x/R$, and we already know that $y \in y/R$, so $y \in x/R \cap y/R$. Therefore $x/R \cap y/R$ is nonempty. Thus we have proved the result.

□

**Definition:** Let $A$ be a set and let $\mathcal{A}$ be a family of subsets of $A$. Set $\mathcal{A}$ is a partition of $A$ iff

i. If $X \in \mathcal{A}$, then $X \neq \emptyset$.

ii. If $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, then $X = Y$ or $X \cap Y = \emptyset$.

iii. $\displaystyle\bigcup_{X \in \mathcal{A}} X = A$.

**Example:** Consider the equivalence relation $R$ on the set $A = \{3, 4, 5\}$. Let $\mathcal{A}$ be the set of distinct equivalence classes. Then

$$\mathcal{A} = \{\{3, 5\}, \{4\}\}$$

is a partition of $A$.

**Note:** The previous theorem says that every equivalence relation $R$ on set $A$ determines equivalence classes which partition set $A$.

**Question:** Is the converse true? Does every partition of set $A$ determine an equivalence relation on $A$?

**Example:** Let $A = \{1, 2, 3, 4\}$ and let $\mathcal{A} = \{\{1, 3\}, \{2\}, \{4\}\}$ be a partition of $A$. Determine an equivalence relation $R$ on $A$ whose equivalence classes are precisely the elements of $\mathcal{A}$.

**Solution:**

$$\begin{aligned} R &= \{(1,1),(1,3),(3,1),(3,3),(2,2),(4,4)\} \\ 1/R &= \{1,3\}, \\ 2/R &= \{2\}, \\ 3/R &= \{1,3\}, \\ 4/R &= \{4\} \end{aligned}$$

# Worksheet Example B

**Theorem:** Let $\mathcal{B}$ be a partition of set $A$. For $x, y \in A$ define relation $R$ on $A$ by "$x\ R\ y$ iff there exists a set $C \in \mathcal{B}$ so that $x \in C$, and $y \in C$." Then

a. $R$ is an equivalence relation on $A$.

b. $A/R = \mathcal{B}$.