Equivalence Relations

Consider the following question:
If it is 8 a.m. now, then what time will it be in 110 hours?
One way to do this is to divide 110 by 24 and look at the remainder.

$$110 = (24)(4) + 14$$

Then 8 a.m. + 14 hours is 10 p.m.

Here we are making 110 hours "equivalent" to 14 hours because they are linked by the following property:
Their difference is divisible by 24.
This linking (or relation) is going to become very useful and important.

Definition: Let A be a set, and let R be a relation on A. (Recall this means $R \subseteq A \times A$.) Then

1. R is *reflexive* iff $x \; R \; x$ for all $x \in A$.

2. R is *symmetric* iff $x \; R \; y \; \Rightarrow y \; R \; x$.

3. R is *transitive* iff $x \; R \; y$ and $y \; R \; z \Rightarrow x \; R \; z$.

Example: Let $A = \{2, 3, 4\}$. Consider the following relations R on A.

1. $R = \{(2, 2), (2, 3), (3, 3), (4, 4)\}$ is reflexive on A.

2. $R = \{(2, 2), (3, 3)\}$ is not reflexive on A.

3. $R = \{(2,3),(3,2),(3,4),(4,3)\}$ is symmetric.

4. $R = \{(3,3),(3,4)\}$ is not symmetric.

5. $R = \{(2,3),(3,3)\}$ is transitive.

6. $R = \{(2,3),(3,4),(4,2),(2,4),(4,3),(3,2)\}$ is not transitive.

Question: If a relation is both transitive and symmetric, does that mean that it must be reflexive?

Worksheet Example A

Definition: A relation R on a set A is an equivalence relation on A iff R is reflexive on A, symmetric, and transitive.

Worksheet Example B

Definition: Let R be an equivalence relation on A and let $x \in A$. The *equivalence class of* x determined by R, written x/R, is the set

$$x/R = \{y \in A \ : \ x \ R \ y\}.$$

The notation x/R is read "x mod R".
The set of all equivalence classes is called A *modulo* R and is denoted by

$$A/R = \{x/R \ : \ x \in A\}.$$

Example: In example B number 1, the relation R is an equivalence relation on set $A = \{3, 4, 5\}$. The equivalence classes determined by R are:

$$\begin{aligned} 3/R &= \{y \in A : 3 \; R \; y\} = \{3, 5\} \\ 4/R &= \{y \in A : 4 \; R \; y\} = \{4\} \\ 5/R &= \{y \in A : 5 \; R \; y\} = \{3, 5\} \end{aligned}$$

So that

$$A/R = \{\{3, 5\}, \{4\}\}.$$

Example: The relation R in example B number 6 is an equivalence relation on set $A = \mathbb{Z}$. The equivalence classes determined by R are:

$$\begin{aligned}
0/R &= \{y \in \mathbb{Z} \ : \ 3 \mid (0 - y)\} \\
&= \{\ldots, -6, -3, 0, 3, 6, \ldots\} \\
1/R &= \{y \in \mathbb{Z} \ : \ 3 \mid (1 - y)\} \\
&= \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\
2/R &= \{y \in \mathbb{Z} \ : \ 3 \mid (2 - y)\} \\
&= \{\ldots, -4, -1, 2, 5, 8, \ldots\} \\
3/R &= \{y \in \mathbb{Z} \ : \ 3 \mid (3 - y)\} \\
&= \{\ldots, -3, 0, 3, 6, 9, \ldots\} \\
-1/R &= \{y \in \mathbb{Z} \ : \ 3 \mid (-1 - y)\} \\
&= \{\ldots, -7, -4, -1, 2, 5, \ldots\} \\
&\vdots
\end{aligned}$$

so that

$$A/R = \{0/R, 1/R, 2/R\}$$

Worksheet Example C