Cartesian Products

Ordered Pairs

An ordered pair (a, b) is just like a list or set, except that **order matters.** In other words though $\{a, b\} = \{b, a\}$, in general $(a, b) \neq (b, a)$.

$$(a, b) = (x, y) \quad \iff a = x \text{ and } b = y.$$

Definition: Let A and B be sets. Then *Cartesian product* of A and B is

$$A \times B = \{(a, b) \ : \ a \in A \ and b \in B\}.$$

Example: Let $A = \{\Delta, 3\}$ and $B = \{r, \{2\}, 7\}$. Then

$$A \times B = \{(\Delta, r), (\Delta, \{2\}), (\Delta, 7), (3, r), (3, \{2\}), (3, 7)\}.$$

Note: In general $A \times B \neq B \times A$. Also note that $(x, y) \in A \times B$ means $x \in A$, and $y \in B$.

Example: Let $A = \{a, b, c\}$. Then

$$A \times A = \\ \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c), \\ (c,a),(c,b),(c,c)\}.$$

Are you familiar with $\mathbb{R} \times \mathbb{R}$? This is just the x, y plane.

Theorem: Let A be any set. Then

1. $A \times \emptyset = \emptyset$.
2. $\emptyset \times A = \emptyset$.

proof: Let A be any set. Let us show that $A \times \emptyset = \emptyset$. To do this, notice that we already know $\emptyset \subseteq A \times \emptyset$, therefore we need only show $A \times \emptyset \subseteq \emptyset$.

To do this notice that $(x, y) \in A \times \emptyset$ means that $x \in A$, and $y \in \emptyset$. But $y \in \emptyset$ is false, so the statement $(x, y) \in A \times \emptyset$ is false. Therefore the statement if $(x, y) \in A \times \emptyset$ then $(x, y) \in \emptyset$ is true. Therefore $A \times \emptyset \subseteq \emptyset$ as desired, and thus $A \times \emptyset = \emptyset$.

The proof of 2) is similar.

$\square$

Theorem: Let A, B, C and D be nonempty sets. Then

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

4. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

proof of 1): Let us prove this using a string of bi-conditionals. So

$$\begin{aligned} & (x,y) \in A \times (B \cup C) \\ \iff & x \in A, \quad \text{and} \quad y \in B \cup C \\ \iff & x \in A \quad \text{and} \quad (y \in B \quad \text{or} \quad y \in C) \\ \iff & (x \in A \quad \text{and} \quad y \in B) \quad \text{or} \quad (x \in A \quad \text{and} \quad y \in C) \\ \iff & (x,y) \in A \times B \quad \text{or} \quad (x,y) \in A \times C \\ \iff & (x,y) \in (A \times B) \cup (A \times C). \end{aligned}$$

Therefore $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

□

proof of 4): Let $(x,y) \in (A \times B) \cup (C \times D)$. Then we have one of two cases, either $(x,y) \in A \times B$, or $(x,y) \in C \times D$.

Case 1: Suppose that $(x, y) \in A \times B$, then $x \in A$, and $y \in B$. Since $x \in A$, then $x \in A \cup C$. Similarly since $y \in B$, then $y \in B \cup D$. Therefore in this case $(x, y) \in (A \cup C) \times (B \cup D)$.

Case 2: Now suppose that $(x, y) \in C \times D$, then $x \in C$, and $y \in D$. Since $x \in C$, then $x \in A \cup C$. Similarly since $y \in D$, then $y \in B \cup D$. Therefore in this case $(x, y) \in (A \cup C) \times (B \cup D)$.

So in both cases we have $(x, y) \in (A \cup C) \times (B \cup D)$. Thus $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

$\square$

Note:

1. All four statements in the previous theorem are true even if at least one of the sets is $\emptyset$.

2. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then

$$\begin{aligned} A \times B &= \{(1,2)\} \\ C \times D &= \{(3,4)\} \\ A \cup C &= \{1,3\} \\ B \cup D &= \{2,4\} \\ (A \cup C) \times (B \cup D) &= \{(1,2),(1,4),(3,2),(3,4)\} \\ (A \times B) \cup (C \times D) &= \{(1,2),(3,4)\} \end{aligned}$$

So the sets clearly are not equal in this case.

Relations

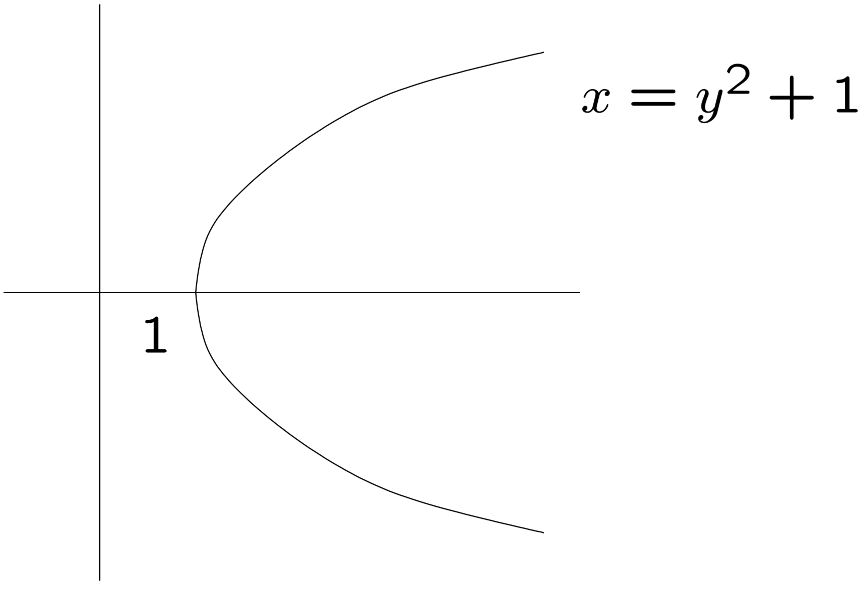
Definition: Let A and B be sets. A *relation* from A to B is a subset of $A \times B$.

If R is a relation from A to B and $(a,b) \in R$, we write $a \; R \; b$. If $(a,b) \notin R$, then we write $a \; \not R \; b$.

Subsets of $A \times A$ are called *relations on* A.

Example: The set $R = \{(\Delta, r), (3, r), (3, 7)\}$ is a relation from A to B as given in the earlier example.

Example: A *graph* is a well-known relation. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \ : \ x = y^2 + 1\}$. Then S is a relation on the set of real numbers $\mathbb{R}$.



If a set A has n elements, and a set B has m elements, then

1. $A \times B$ has mn elements.

2. There are 2^{mn} relations from A to B (subsets of $A \times B$).

Example: If $A = \{p, q, r\}$ and $B = \{3, 7\}$, then $A \times B$ has (3)(2) $= 6$ elements and there are $2^6 = 64$ relations from A to B.

Definition: Let R be a relation from A to B.

1. The *domain* or R is

$$\text{Dom}(R) = \{x \in A \ : \ \exists y \in B \text{ with } x \ R \ \}.$$

2. The *range* of R is

$$\text{Rng}(R) = \{y \in B \ : \ \exists x \in A \text{ with } x \ R \ \}.$$

Note: Clearly, Dom(R) $\subseteq A$ and Rng(R) $\subseteq B$ since $R \subseteq A \times B$.

Worksheet Example A

Definition: If R is a relation from A to B, then the *inverse* of R is

$$R^{-1} = \{(y, x) \ : \ (x, y) \in R\}.$$

Example: If relation $R = \{(a, b), (a, c), (c, d)\}$, then its inverse relation is

$$R^{-1} = \{(b, a), (c, a), (d, c)\}.$$

Worksheet Example B

Theorem: Let R be a relation from A to B, (i.e. $R \subseteq A \times B$). Then

1. R^{-1} is a relation from B to A.
2. $\mathrm{Dom}(R^{-1}) = \mathrm{Rng}(R)$.
3. $\mathrm{Rng}(R^{-1}) = \mathrm{Dom}(R)$.

proof of 1): Assume that $R \subseteq A \times B$. Let $(r, s) \in R^{-1}$. Then this gives us that $(s, r) \in R$, and since $R \subseteq A \times B$, then $(s, r) \in A \times B$ as well. However this means that $s \in A$, and $r \in B$. So that $(r, s) \in B \times A$. Therefore we have shown that $R^{-1} \subseteq B \times A$ and thus R^{-1} is a relation from B to A.

$\square$

Definition: Let R be a relation from A to B, and let S be a relation from B to C. The *composite* of R and S, written $S \circ R$, is the set
$S \circ R =$
$\{(a, c) \ : \ (\exists b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$.

Note:

1. $S \circ R \subseteq A \times C$ so $S \circ R$ is a relation from A to C.
2. Dom$(S \circ R) \subseteq$ Dom(R) but they are not always equal.
3. In general, $S \circ R \neq R \circ S$.

Worksheet Example C