Equivalent Forms of Induction

We now introduce the Principle of Complete Induction (PCI) along with the Well-Ordering Principle (WOP) both equivalent to the Principle of Mathematical Induction (PMI).

I. Principle of Mathematical Induction (PMI) - To prove $(\forall n \in \mathbb{N})P(n)$:

1. Show $P(1)$ is true.

2. Assume that for some $k \in \mathbb{N}$ $P(k)$ is true.

3. Show that $P(k+1)$ is true.

4. Thus $P(n)$ is true $\forall n \in \mathbb{N}$.

II. Principle of Complete Induction (PCI) -
To prove $(\forall n \in \mathbb{N})P(n)$:

1. Assume that for some fixed $m \in \mathbb{N}$ $P(k)$ is true $\forall k \in \{1, 2, 3, \ldots, m-1\}$.

2. Show that $P(m)$ is true.

3. Thus, $P(n)$ is true $\forall n \in \mathbb{N}$.

III. Well-Ordering Principle (WOP) -
If $S \neq \emptyset$ and $S \subseteq \mathbb{N}$, then S has a smallest element.

It can be proven that these three principles are equivalent.

Some observations:

Consider the set $S = \{1, 2, 3, \ldots, m-1\}$ for each $m \in \mathbb{N}$. Then when $m = 1$, then $S = \emptyset$, when $m = 2$, $S = \{1\}$, when $m = 3$, $S = \{1, 2\}$ and so on.

Recall: A natural number $p > 1$ is *prime* if 1 and p are its only fators in $\mathbb{N}$. Otherwise, p is *composite*.

Example: Prove that every natural number $n > 1$ has a prime factor.
proof: (Proof by PCI) Assume that there is some fixed $m \in \mathbb{N}$ such that if $k \in \{2, 3, \ldots, m-1\}$, then k has a prime factor. Since $m = 2$ has a prime factor, then we may assume $m \geq 3$. We now show that m has a prime factor. Consider the following cases:
Case 1: Assume m is prime. In this case since m is a factor of itself, then m has a prime

factor.

Case 2: Assume m is composite. Then there is some integer d with $1 < d < m$ and $d \mid m$. Since $d \in \{2, 3, 4, \ldots, m-1\}$, then d must have a prime factor p by assumption. But since $p \mid d$, and $d \mid m$, then $p \mid m$. Therefore m has a prime factor as well.

Therefore by the PCI we have shown that every natural number $n > 1$ has a prime factor.

$\square$

Notice that the regular PMI would not have proved this this easily.

Fibonacci Numbers

Fibonacci Numbers are those listed in the following sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Let f_n be the nth Fibonacci number for $n = 1, 2, 3, \ldots$. Then

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 1 \\ f_{n+2} &= f_{n+1} + f_n \end{aligned}$$

Example: Assume that f_n is a Fibonacci number $\forall n \in \mathbb{N}$. Show that

$$f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$$

for all $n \in \mathbb{N}$.
proof: (Using the PCI) Assume that there is

some fixed $m \in \mathbb{N}$ such that if $k \in \{1, 2, 3, \ldots, m-1\}$, then

$$f_1 + f_2 + f_3 + \cdots + f_k = f_{k+2} - 1.$$

Now we must show that the the result holds for m. Note that $f_1 = f_3 - 1$, so assume that $m \geq 2$. Then

$$\begin{aligned} & f_1 + f_2 + \cdots + f_{m-1} + f_m \\ = & \ (f_1 + f_2 + \cdots + f_{m-1}) + f_m \\ = & \ (f_{m-1+2} - 1) + f_m \\ = & \ f_{m+1} + f_m - 1 \\ = & \ f_{m+2} - 1 \end{aligned}$$

Therefore

$$f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$$

for all $n \in \mathbb{N}$.

$\square$

Example: Prove that every natural number $n > 1$ has a prime factor.

proof: (Using the WOP) Let n be a natural number with $n > 1$. Consider the following two cases:

Case 1: Assume n is prime. Then n has itself as a prime factor.

Case 2: Assume that n is composite. Then there is an integer d so that $1 < d < n$ and $d \mid n$. Let

$$T = \{x \in \mathbb{N} \ : \ 1 < x < n \text{ and } x \mid n\}.$$

Note that $T \subseteq \mathbb{N}$, and since $d \in T$, then $T \neq \emptyset$. Thus, by the WOP this set T must have a smallest element p. We claim that p is a prime. Suppose that p is not a prime, then p is composite. Since $p > 1$, then there is an integer j such that $1 < j < p$, and $j \mid p$. However, if $1 < j < n$, then $1 < j < x$. Also, if $j \mid p$, then since $p \mid n$, then $j \mid n$. Therefore $j \in T$. However $j < p$, and this contradicts the fact that p is the smallest element of T. Therefore p must not be composite, and therefore must be prime. Therefore n has a prime factor as desired.

$\square$

Example A on Worksheet

The following two theroems are proved using the Well-Ordering Principle.

Theorem: (Division Algorithm for $\mathbb{N}$.) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ with $b \leq a$. Then there exists $q \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ so that

$$a = qb + r,$$

where $0 \leq r < b$.

Theorem: (Greatest Common Divisor) Let $a, b \in \mathbb{N}$ and d be the greatest common divisor of a and b. Then there exists integer x and y so that $ax + by = d$.

Definition: If GCD$(a, b) = 1$, then we say that a and b are *relatively prime.*

The Equivalence of PMI, PCI, and WOP

I. PMI: Let $S \subseteq \mathbb{N}$. Assume that

1. $1 \in S$. and

2. $(\forall n \in \mathbb{N})(n \in S \Rightarrow n + 1 \in S)$.

Then $S = \mathbb{N}$.

II. PCI: Let $S \subseteq \mathbb{N}$. Assume that

$$(\forall m \in \mathbb{N})(\{1, 2, 3, \ldots, m-1\} \subseteq S \Rightarrow m \in S).$$

Then $S = \mathbb{N}$.

III. WOP: If $T \subseteq \mathbb{N}$ and $T \neq \emptyset$, then T has a smallest element.

Let us prove that I is equivalent to III.

PMI $\Rightarrow$ WOP:
proof: Assume that PMI is true. Let $T \subseteq \mathbb{N}$ and assume that $T \neq \emptyset$. Let us show that T has a smallest element. Suppose that it does not. Let

$$S = N - T = N \cap \widetilde{T}.$$

If $1 \in T$, then 1 would be the smallest element in T. Thus $1 \in S$. Assume that $k \in S$ for some $k \in \mathbb{N}$. We show that $k+1 \in S$. To do this consider if $k+1 \in T$, then one of $2, 3, 4, \ldots, k-$

$1, k+1$ is the least element in T. But T does not have a smallest element, so $k+1 \in S$. Thus by PMI $S = \mathbb{N}$. But if $S = \mathbb{N}$, then $T = \emptyset$. This is a contradiction. Thus T has a smallest element.

$\square$

WOP $\Rightarrow$ PMI:
proof: Assume WOP is true. Assume $S \subseteq \mathbb{N}$ with the properties

1. $1 \in S$ and

2. $(\forall n \in \mathbb{N})(n \in S \Rightarrow n+1 \in S)$.

We will shoe that $S = \mathbb{N}$. Suppose not, suppose that $S \neq \mathbb{N}$. Let

$$T = N - S = N \cap \widetilde{S}.$$

Then $T \subseteq \mathbb{N}$, and since $S \neq \mathbb{N}$, then $T \neq \emptyset$. So by WOP T has a smallest element $m \in T$. Consider the element $m - 1$. Since m is the smallest element in T and $m - 1 < m$, then $m - 1 \in S$. But if $m - 1 \in S$, then $m - 1 + 1 = m \in S$ as well. But $m \in T$ and $S \cap T = \emptyset$. This is a contradiction. Thus, $S = \mathbb{N}$.

$\square$