More on Sets of Sets

Definition: A set of sets if often called a *family of sets*.

Example:

$$S = \{\{1, 2\}, \{2, \sqrt{2}\}, \{3, 1, 7, 6\}\}$$

Definition: Let $\mathcal{A}$ be a family of sets. The *union over* $\mathcal{A}$ is defined by

$$\bigcup_{A \in \mathcal{A}} A = \{x \ : \ (\exists A \in \mathcal{A})(x \in A)\}$$

and the *intersection over* $\mathcal{A}$ is

$$\bigcap_{A \in \mathcal{A}} A = \{x \ : \ (\forall A \in \mathcal{A})(x \in A)\}$$

Example: Let $\mathcal{A} = \{\{1\}, \{1, 2\}, \{2, 3\}\}$. Then

$$\bigcup_{A \in \mathcal{A}} A = \{1, 2, 3\} \quad \text{and} \quad \bigcap_{A \in \mathcal{A}} A = \emptyset.$$

Another Way

Definition: Let Δ be a nonempty set. Suppose for each $\alpha \in \Delta$, there is a corresponding set A_α. Then the family of sets

$$\mathcal{A} = \{A_\alpha \ : \ \alpha \in \Delta\}$$

is an *indexed family of sets.* Each $\alpha \in \Delta$ is called an *index*, and Δ is called an *indexing set.*

Example: Let $\Delta = \mathbb{N}$. Let

$$I_1 = [0,1], \ I_2 = \left[0, \frac{1}{2}\right], \ I_3 = \left[0, \frac{1}{3}\right], \ldots$$

What is $\bigcup_{n \in \mathbb{N}} I_n$?

Answer:

$$\bigcup_{n \in \mathbb{N}} I_n = [0,1].$$

What is $\bigcap_{n\in\mathbb{N}} I_n$?

Answer:

$$\bigcap_{n\in\mathbb{N}} I_n = \{0\}.$$

What about

$$I_1 = (0,1),\ \ I_2 = \left(0, \frac{1}{2}\right),\ \ I_3 = \left(0, \frac{1}{3}\right), \ldots \quad ?$$

Sometimes we will use the following notation:

$$\bigcup_{i=1}^{4} I_i = I_1 \cup I_2 \cup I_3 \cup I_4$$

and

$$\bigcap_{i=3}^{6} I_i = I_3 \cap I_4 \cap I_5 \cap I_6.$$

Example: Let $A_n = [0, n)$ for each $n \in \mathbb{N}$. and let

$$\begin{aligned} \mathcal{A} &= \{A_n \ : \ n \in \mathbb{N}\} \\ &= \{[0,1), [0,2), [0,3), [0,4), \ldots\}. \end{aligned}$$

Then

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{i=1}^{\infty} A_i = [0, \infty)$$

and

$$\bigcap_{A \in \mathcal{A}} A = \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{i=1}^{\infty} A_i = [0, 1).$$

Example: Let $A_r = \{1, r^3\}$ for each $r \in \mathbb{R}$ and let $\mathcal{A} = \{A_r \ : \ r \in \mathbb{R}\}$. Then

$$\bigcap_{A \in \mathcal{A}} A = \bigcap_{r \in \mathbb{R}} A_r = \{1\},$$

and

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{r \in \mathbb{R}} A_r = \mathbb{R},$$

Worksheet Example A

Some Theorems

Theorem: Let $\mathcal{A}$ be a family of sets and let $B \in \mathcal{A}$. Then

1. $\bigcap_{A \in \mathcal{A}} A \subseteq B$.

2. $B \subseteq \bigcup_{A \in \mathcal{A}} A$.

proof:

1) Assume that $\mathcal{A}$ is a family of sets and that $B \in \mathcal{A}$. Now let $x \in \bigcap_{A \in \mathcal{A}} A$, Then $x \in A$ for all sets $A \in \mathcal{A}$. But $B \in \mathcal{A}$, so $x \in B$. Thus we have shown that $\bigcap_{A \in \mathcal{A}} A \subseteq B$.

2) Exercise.

$\square$

Theorem: Let $\mathcal{A} = \{A_i \,:\, i \in \mathbb{N}\}$ be an indexed family of sets, and let m be a fixed integer with $m \geq 1$. Then

1. $\displaystyle\bigcup_{i=1}^{m} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

2. $\displaystyle\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=1}^{m} A_i$.

proof: Assume the hypothesis.

1) Let $x \in \displaystyle\bigcup_{i=1}^{m} A_i$, then x is an element of at least one of the sets $A_1, A_2, \ldots, A_m$, or said differently $x \in A_k$ for some k where $1 \leq k \leq m$. But $A_k \in \mathcal{A}$ so

$$x \in \bigcup_{A \in \mathcal{A}} A = \bigcup_{i=1}^{\infty} A_i.$$

Therefore $\bigcup_{i=1}^{m} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

2) Now let $x \in \bigcap_{i=1}^{\infty} A_i$. Then $x \in A_i$ for all $i = 1, 2, 3, 4, \ldots$. In particular $x \in A_i$ for $i = 1, 2, 3, 4, \ldots, m$, so that

$$x \in \bigcap_{i=1}^{m} A_i.$$

Therefore $\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=1}^{m} A_i$.

$\square$

Theorem: Let $\mathcal{A} = \{A_\alpha \ : \ \alpha \in \Delta\}$ be an indexed family of sets. Then

1. $\widetilde{\bigcup_{\alpha \in \Delta} A_\alpha} = \bigcap_{\alpha \in \Delta} \widetilde{A_\alpha}$.

2. $\widetilde{\bigcap_{\alpha\in\Delta} A_\alpha} = \bigcup_{\alpha\in\Delta} \widetilde{A_\alpha}$.

proof:

1)

$$
\begin{aligned}
x \in \widetilde{\bigcup_{\alpha\in\Delta} A_\alpha} &\iff x \notin \bigcup_{\alpha\in\Delta} A_\alpha \\
&\iff x \notin A_\alpha \text{ for all } \alpha \in \Delta \\
&\iff x \in \widetilde{A_\alpha} \text{ for all } \alpha \in \Delta \\
&\iff x \in \bigcup_{\alpha\in\Delta} \widetilde{A_\alpha}.
\end{aligned}
$$

(Note that we have shown the equivalent of $\widetilde{\bigcup_{\alpha\in\Delta} A_\alpha} \subseteq \bigcap_{\alpha\in\Delta} \widetilde{A_\alpha}$ and $\bigcap_{\alpha\in\Delta} \widetilde{A_\alpha} \subseteq \widetilde{\bigcup_{\alpha\in\Delta} A_\alpha}$.)

2) Exercise.

□

Definition: The family of sets $\mathcal{A} = \{A_\alpha \ : \ \alpha \in \Delta\}$ is *pairwise disjoint* iff for all $\alpha, \beta \in \Delta$, if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \emptyset$.

Example: Each of the following families of sets is pairwise disjoint.

1. $\mathcal{A} = \{\{1\}, \{2, 3\}, \{5, 7, 10\}\}$

2. $\mathcal{A} = \{(n, n+1] \ : \ n \in \mathbb{N}$

Definition: A family of sets $\mathcal{A} = \{A_i \ : \ i \in \mathbb{N}\}$ is a *nested* family of sets if for all $i, j \in \mathbb{N}$ with $i \leq j$, then $A_j \subseteq A_i$.

Example: Each of the following families of sets is a nested family of sets.

1. Let $A_n = \left[2, 2 + \frac{1}{n}\right]$ for each $n \in \mathbb{N}$ and let $\mathcal{A} = \{A_n \ : \ n \in \mathbb{N}\}$.

2. Let $A_n = (n + 3, \infty)$ for each $n \in \mathbb{N}$ and let $\mathcal{A} = \{A_n \ : \ n \in \mathbb{N}\}$.