Sets

Definition: A *set* is a specified collection of objects.

Definition: The objects in a given set are called *elements (or memebers).*

Sets are described using $\{\quad\}$.

Sometimes we simply list every element:

$$K = \{5, 6, \text{ a white cloud, an old desk}\}$$

Sometimes we give an indication of what they look like:

$$\begin{aligned} J &= \{2, 4, 6, 8, \ldots\} \\ L &= \{\ldots, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \end{aligned}$$

Or more importantly (usefully) in set-builder notation:

$$\{x \ : \ P(x)\}$$

where $P(x)$ is a predicate that we call the *rule.*

Example:

$$\{x \in \mathbb{R} \ : \ x \geq 0\} = [0, \infty)$$

Some Common Notation:

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
&= \text{The Natural Numbers} \\
\mathbb{Z} &= \{\ldots, -3, -2. - 1.0, 1, 2, 3, \ldots\} \\
&= \text{The Integers} \\
\mathbb{Q} &= \text{The Set of Rational Numbers} \\
\mathbb{R} &= \text{The Set of Real Numbers} \\
(a, b) &= \{x \ : \ x \in \mathbb{R} \text{ and } a < x < b\} \\
&= \text{The Open Interval from } a \text{ to } b. \\
[a, b] &= \{x \in \mathbb{R} \ : \ a \leq x \leq b\} \\
&= \text{The Closed interval from } a \text{ to } b.
\end{aligned}$$

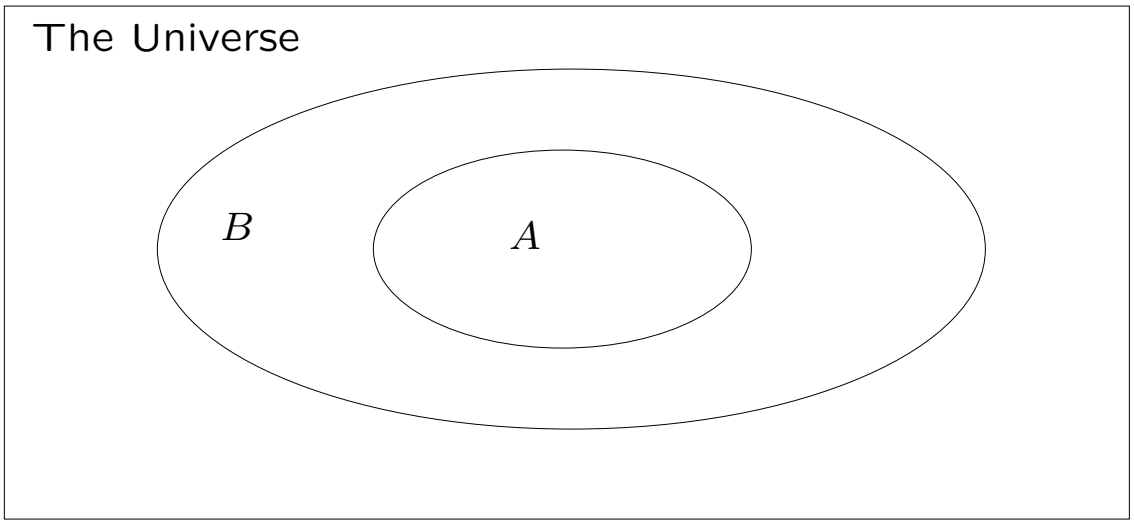
Definition: Let $\emptyset = \{x \ : \ x \neq x\}$. Then $\emptyset$ is a set with no elements and is called an *empty set.*

$x \in \emptyset$ is false for every object x.

Definition: Let A and B be sets. We say that A is a *subset* of B iff every element of A is also an element of B. In symbols,

$$A \subseteq B \iff (\forall x)(x \in A \Rightarrow x \in B).$$

Venn Diagrams



Direct Proof of $A \subseteq B$

Technique:

Let x be any object
$\langle$ Show that $x \in A \Rightarrow x \in B \rangle$
Consider $A = \emptyset$ case (Vacuous).
Then Suppose $x \in A$
$\vdots$
Thus $x \in B$
Therefore $A \subseteq B$.

Theorem: For any set A, $\emptyset \subseteq A$.
proof: Let x be any object,

$$(x \in \emptyset \Rightarrow x \in A)$$

is a true statement since $x \in \emptyset$ is false. Therefore $\emptyset \subseteq A$.

$\square$

Theorem: For any set A, $A \subseteq A$.
proof: Let x be any object. Suppose that $x \in A$, then $x \in A$. Thus $A \subseteq A$.

□

Maybe a little more interesting

Theorem: Let A, B, and C be sets. If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.
proof: Suppose $A \subseteq B$ and $B \subseteq C$. If $A = \emptyset$, then the result holds. So suppose $A \neq \emptyset$, and let $x \in A$. Then since $A \subseteq B$, then $x \in B$. But since $B \subset C$, then $x \in C$. Thus $A \subseteq C$.

□

$$A = B \iff (\forall x)(x \in A \iff x \in B)$$

$$(\forall x)(x \in A \Rightarrow x \in B) \wedge (\forall x)(x \in B \Rightarrow x \in A)$$

Definition: Let A and B be sets. Then $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

To prove $A = B$

Technique:

Prove $A \subseteq B$
Prove $B \subseteq A$
Conclude $A = B$.

Theorem: The empty set is unique.
proof: Let A and B be two sets with no elements. Then the statement

$$(\forall x)(x \in A \Rightarrow x \in B)$$

is true, so $A \subseteq B$. Similarly, the statement

$$(\forall x)(x \in B \Rightarrow x \in A)$$

is true, so $B \subseteq A$. Therefore $A = B$ and so there is only one empty set.

Sets of Sets

Example: Consider

$$J = \{A, B, \{1, 2, 4\}, \{1, 4\}\}$$

Is $1 \in J$?

If $a \in A$, is $a \in J$?

Is $\{1\} \subseteq J$?

Is $\{1\} \in J$?

Is $\{1, 4\} \subseteq J$?

Is $\{1, 4\} \in J$?

Example A on Worksheet

A Special Set of Sets

Definition: Let A be a set. The *power set of* A is the set whose elements are the subsets of A, and is denoted $\mathcal{P}$.

Thus $\mathcal{P} = \{B \ : \ B \subseteq A\}$.

Example: Let $L = \{1, 2, 7\}$
Subsets of L:

$$\{1\}, \{2\}, \{7\}, \{1, 2\}, \{1, 7\}, \{2, 7\}, L, \emptyset$$

So

$$\mathcal{P}(L) = \{\{1\}, \{2\}, \{7\}, \{1, 2\}, \{1, 7\}, \{2, 7\}, L, \emptyset\}$$

Note that L has 3 elements, and $\mathcal{P}$ has 8 elements.

The Rest of the Examples on the Worksheet

Theorem: If set A has n elements, then the power set $\mathcal{P}(A)$ has 2^n elements.
proof: See text.

Theorem: Let A and B be sets. Then $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
proof: ...

Notice:
$\{\emptyset\}$ is different from $\emptyset$. (Why?)