Proof by Contrapositive

Tautologies can be used at any time in a proof. So proving $P \Rightarrow Q$ is the same as proving $\sim Q \Rightarrow \sim P$.

Technique:

$$\begin{gathered} \text{Assume } \sim Q \\ \vdots \\ \text{Show } \sim P, \\ \text{Then } \sim Q \Rightarrow \sim P \\ \text{or } P \Rightarrow Q. \end{gathered}$$

Example: If n^2 is an odd integer, then n is odd.
proof:
Suppose n is an even integer ($\sim Q$). Then there exists and integer w such that $n = 2w$. Thus

$$n^2 = (2w)^2 = 4w^2 = 2(2w^2).$$

Since $2w^2$ is an integer, then n^2 is even. Therefore n^2 is not odd ($\sim P$).

$\square$

Example: For all real numbers x, if $x^3 + x^2 - 2x < 0$, then $x < 1$.
(Scratch Work)
Suppose that $x \geq 1$ ($\sim Q$). (We wish to show ($\sim P$) that $x^3 + x^2 - 2x \geq 0$). To do this we work backwards giving us the following:

$$\begin{aligned} x^3 + x^2 - 2x &\geq 0 \\ x(x^2 + x - 2) &\geq 0 \\ x(x-1)(x+2) &\geq 0 \end{aligned}$$

This is true when $x \geq 1$, so we are ready for our proof.
proof:
Suppose that $x \geq 1$, then $x \geq 0$, $(x - 1) \geq 0$, and $(x + 2) \geq 0$. So

$$\begin{aligned} x(x-1)(x+2) &\geq 0 \quad \Rightarrow \\ x(x^2 + x - 2) &\geq 0 \quad \Rightarrow \\ x^3 + x^2 - 2x &\geq 0. \end{aligned}$$

Therefore if $x \geq 1$, then $x^3 + x^2 - 2x \geq 0$. Thus if $x^3 + x^2 - 2x < 0$, then $x < 1$.

$\square$

Example 1 on Worksheet

Construct a truth table for $(\sim Q) \Rightarrow (P \wedge \sim P)$.

P	Q	$\sim Q$	$\sim P$	$P \wedge \sim P$	$\sim Q \Rightarrow (P \wedge \sim P)$
T	T	F	F	F	T
T	F	T	F	F	F
F	T	F	T	F	T
F	F	T	T	F	F

So Q is equivalent to $\sim Q \Rightarrow (P \wedge \sim P)$.

Replacing Q with $A \Rightarrow B$ gives us

$$\begin{aligned} (A \Rightarrow B) &\equiv (\sim (A \Rightarrow B) \Rightarrow (P \wedge \sim P)) \\ &\equiv ((A \wedge \sim B) \Rightarrow (P \wedge \sim P)) \end{aligned}$$

Proof by Contradiction

Technique:

Suppose A and Suppose B.

$\vdots$

Therefore P.

$\vdots$

Therefore $\sim P$.

So $P \wedge \sim P$, a Contradiction.

Thus $A \Rightarrow B$.

Example: Suppose a is a real number. Show that if $a > 0$, then $\frac{1}{a} > 0$.

proof:

Assume that $a > 0$ (Assume A), and suppose that $\frac{1}{a} \leq 0$ (Suppose $\sim B$). Since $\frac{1}{a} \leq 0$, there is some number $b \geq 0$ such that $\frac{1}{a} + b = 0$. Multiplying both sides by a gives us $1 + ab = 0$. Since $a > 0$, and $b \geq 0$, $ab \geq 0$, so $1 \leq 0$ (P).

But we know that $1 > 0$ ($\sim P$), so this is a contradiction. So our assumption that $\frac{1}{a} \leq 0$ must have been false, so $\frac{1}{a} > 0$. Thus the result If $a > 0$, then $\frac{1}{a} > 0$ is true.

$\square$

Correct Mistake in Definition

Definition: A natural number (positive integer) $\mathbf{p > 1}$ is said to be *prime* if the only integers that divide it are 1 and p.

Example: Prove that there are an infinite number of primes.

Notice that this is not an "if, then" statement.

So we use the equivalence of Q and $\sim Q \Rightarrow (P \wedge \sim P)$.

proof:
Suppose that there are a finite number of primes ($\sim Q$). So we can let $p_1, p_2, \ldots, p_n$ be the complete list of all prime numbers.
Now let

$$N = p_1 p_2 \cdots p_n + 1.$$

N is an integer, and N is not prime because it is larger than all the p_i's. Suppose that the integer d, with $1 < d < N$, divides N.
Let p be a prime that divides d. Then since $p \mid d$ and $d \mid N$, then $p \mid N$ (Why? Prove this!), and since p is a prime $p \mid (p_1 p_2 \cdots p_n)$. Therefore p divides $N - p_1 p_2 \cdots p_n$ (why?). But

$$N - p_1 p_2 \cdots p_n = 1.$$

So p divides 1 (P). But since p is prime, $p \neq 1$ ($\sim P$). Thus there are infinitely many primes (Q).

□

Some Needed Facts and Definitions

Theorem: (Fundamental Theorem of Algebra) Every natural number can be written uniquely as the product of prime numbers.

Definition: A real number r is said to be *rational* if r can be written as

$$r = \frac{m}{n}$$

for some integers m and n, with $n \neq 0$.

Definition: A real number is said to be *irrational* if it is not rational.

Example 2 on Worksheet

Proofs of Biconditionals

$$(P \iff Q) \equiv (P \Rightarrow Q) \land (Q \Rightarrow P)$$

See truth table

Technique:

Prove that $P \Rightarrow Q$ in any way.
Prove that $Q \Rightarrow P$ in any way.
Thus $P \iff Q$.

Example: Prove that an integer n is even iff n^2 is even.
proof:
($\Rightarrow$) Assume that n is even, then $n = 2k$ for some integer k. Therefore we have

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $2k^2$ is an integer, then n^2 is even.
($\Leftarrow$) Suppose that n is odd, then $n = 2k + 1$ for some integer k. Therefore we have

$$n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1.$$

Since $2k^2+2k$ is an integer, then n^2 is odd. So by contrapositive, we have shown the desired result.

$\square$

Example 3 on Worksheet