

What is a Proof?

Book Definition: A *proof* is a logically valid deduction of a Theorem.

We use **Axioms** or **Postulates** to prove **Theorems**.

Axioms cannot be proved, they are statements that we begin with and assume are true.

There is also a set of **Undefined Terms**. For these we need definitions.

Idea of what a proof is: A proof is a logically valid argument ending with some statement that we wish to verify.

Proofs are written differently for different people.

Your proofs should be rigorous enough that I am convinced that you could provide every detail if I were to ask you to.

A Couple of Tips

Avoid phrases "obviously", "clearly", or "it is easy to see that" and so on.

If something follows from an **earlier** result, you need to state what result you are using.

A Few General Rules and Pieces of Advice

1. Tautologies can be used at any time in a proof.
2. Pay very close attention to details in definitions. (A vague understanding of a definition is worthless.)
3. Work backwards if you need to, then write the proof. (Presenting and discovering a proof are two different things. You do not always have to tell your audience exactly how you discovered your proof.)
4. **Always** use only **full sentences** and proper grammar in your proofs.

Definitions

1. An integer m is said to be *even* if $m = 2n$ for some integer n .
2. An integer m is said to be *odd* if $m = 2n + 1$ for some integer n .
3. Suppose a and b are integers. We say a *divides* b , written $a \mid b$, if $b = am$ for some integer m .
4. A natural number (positive integer) $p > 1$ is said to be *prime* if the only numbers that divide it are 1 and p .
5. The *absolute value* of a real number z , written $|z|$, is defined in the following way:
 $|z| = z$ if $z \geq 0$:, and $|z| = -z$ if $z < 0$.

Direct Proofs

To prove $P \Rightarrow Q$ directly, we implement the following technique:

- Assume P is true.
- Show Q is true. Use:
 - Definitions
 - Axioms
 - Previously proved theorems
- Thus $P \Rightarrow Q$ is true.
- \square QED (“quod est demonstratum” “which is proven or demonstrated”)

Example: Prove that if $x \mid a$ and $x \mid b$, then $x \mid (a - b)$.

proof: Assume that $x \mid a$ and $x \mid b$. Thus $a = xm$ and $b = xn$ for some integers m and n . Then

$$\begin{aligned} a - b &= xm - xn \\ &= x(m - n) \\ &= xq, \end{aligned}$$

where $q = m - n$ is an integer. Therefore $x \mid (a - b)$.

□

Example: Prove that if m is even and n is odd, then $m + n - 2$ is odd.

proof: Assume that m is even and n is odd. Then $m = 2k$ and $n = 2p + 1$ for some integers k and p . Then

$$m + n - 2 = 2k + (2p + 1) - 2$$

$$\begin{aligned} &= 2(k + p) - 1 \\ &= 2(k + p) - 1 + 2 - 2 \\ &= 2(k + p) - 2 + 1 \\ &= 2(k + p - 1) + 1 \\ &= 2q + 1 \end{aligned}$$

where $q = k + p - 1$ is an integer. Therefore $m + n - 2$ is odd.

□

Example 1 on Worksheet

Proof By Cases

- Begin with a clear written statement of the given facts or assumptions.
- Determine what is is that you are trying to prove.

- Determine all possible cases which must be considered in order to prove the mathematical statement.
- For each case provide a clear argument leading to the result.

Example: Prove that if a and b are real numbers, where $b \neq 0$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

proof: Assume that a and b are real numbers and $b \neq 0$. We will consider the following cases:

- Case 1: Assume that $a \geq 0$ and $b > 0$, so that $\frac{a}{b} \geq 0$. Then $|a| = a$, $|b| = b$, and

$$\left|\frac{a}{b}\right| = \frac{a}{b} = \frac{|a|}{|b|}.$$

- Case 2: Assume that $a \geq 0$ and $b < 0$ so that $\frac{a}{b} \leq 0$. Then $|a| = a$, $|b| = -b$, and

$$\left| \frac{a}{b} \right| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}.$$

- Case 3: Assume that $a < 0$ and $b > 0$, so that $\frac{a}{b} < 0$. Then $|a| = -a$, $|b| = b$, and

$$\left| \frac{a}{b} \right| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}.$$

- Case 4: Assume that $a < 0$ and $b < 0$, so that $\frac{a}{b} > 0$. Then $|a| = -a$, $|b| = -b$, and

$$\left| \frac{a}{b} \right| = \frac{a}{b} = \frac{-a}{-b} = \frac{|a|}{|b|}.$$

Thus, for all possible cases, it has been proven that $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$.

□

Example: Prove that $x + |x - 7| \geq 7$ for all real numbers x .

proof: Assume that x is a real number. We will consider the following cases:

- Case 1: Assume that $x \geq 7$. Then $x - 7 \geq 0$ and $|x - 7| = x - 7$, so that

$$x + |x - 7| = x + (x - 7) = 2x - 7 \geq 2(7) - 7 = 7.$$

- Case 2: Assume that $x < 7$. Then $x - 7 < 0$ and $|x - 7| = -(x - 7) = 7 - x$, so that

$$x + |x - 7| = x + (7 - x) = 7 \geq 7.$$

Thus, for all possible cases, it has been proven that $x + |x - 7| \geq 7$.

□

Example 2 on Worksheet

Proof By Working Backward

- Begin with a clear written statement of the given facts or assumptions.
- Determine what it is you are trying to prove.
- Now using what you are trying to prove, try to work your way back to your assumptions. Remember, you are not proving the converse of your statement, nor is the work that you do a proof. You are simply trying to discover the proof in order to write it up formally.
- Once you have seen the key steps, write the proof formally and in the right direction.

Example: Prove that $x^2 + y^2 \geq 2xy$ for all real numbers x and y .

Scratch work (Not to be included in the proof):

$$\begin{aligned}x^2 + y^2 &\geq 2xy && \iff \\x^2 + y^2 - 2xy &\geq 0 && \iff \\x^2 - 2xy + y^2 &\geq 0 && \iff \\(x - y)^2 &\geq 0\end{aligned}$$

Is this last statement true for all real numbers?

(Yes) So now write the proof.

proof: Assume that x and y are real numbers. Then notice that $(x - y)^2 \geq 0$. Using this we have

$$\begin{aligned}(x - y)^2 &\geq 0 && \Rightarrow \\x^2 - 2xy + y^2 &\geq 0 && \Rightarrow \\x^2 + y^2 &\geq 2xy\end{aligned}$$

as desired.

□

Example 3 on Worksheet