Conditionals

Definition: Let P and Q be propositions. The *conditional sentence* $P \Rightarrow Q$ (read "P implies Q" or "If P, then Q") is true whenever Q is true or P is false.

P is called the *antecedent*.

Q is called the *consequent.*

The possible truth values for $P \Rightarrow Q$ are given in the following truth table:

P	Q	P $\Rightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

An implication in which the hypothesis is false is often said to be *vacuously true.*

Notice that the truth table for $P \Rightarrow Q$ is equivalent to that of $(\sim P) \vee Q$.

P	Q	$\sim$ P	($\sim$ P) $\vee$ Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Biconditionals

Definition: Let P and Q be propositions. The *biconditional sentence* $P \iff Q$ (read "P if and only if Q") is true exactly when P and Q have the same truth values.

P	Q	P $\iff$ Q
T	T	T
T	F	F
F	T	F
F	F	T

Definition: Let P and Q be propositions.
The *converse* of P $\Rightarrow$ Q is Q $\Rightarrow$ P.
The *contrapositive* of P $\Rightarrow$ Q is ($\sim$ Q) $\Rightarrow$ ($\sim$ P).

Theorem: Let P and Q be propositions.
$P \Rightarrow Q$ is equivalent to $(\sim Q) \Rightarrow (\sim P)$.
$P \Rightarrow Q$ is NOT equivalent to $Q \Rightarrow P$.

Proof:
Use truth tables.

Example: The following statements are equivalent:
If a function is differentiable at x, then the function is continous at x.

If a function is not continous at x, then the function is not differentiable at x.

Example: The following statements are not equivalent:
If I go to Mexico and lay on the beach, then I will get a suntan.

If I have a suntan, then I went to Mexico and laid on the beach.

Theorem: Let P, Q and R be propositions.

- $P \iff Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

- $\sim (P \wedge Q)$ is equivalent to $(\sim P) \vee (\sim Q)$.

- $\sim (P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$.

- $P \Rightarrow Q$ is equivalent to $Q \vee (\sim P)$.

- $P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$.

- $P \wedge (Q \wedge R)$ is equivalent to $(P \wedge Q) \wedge R$.

- $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.

- $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$.

Proof:
One can see the equivalence of these using truth tables.

Note: We will assume that the statement $P \Rightarrow Q$ (If P, then Q) is equivalent to EACH of the following statements.

- Q if P.

- P only if Q.

- P only when Q.

- Q whenever P.

- Q when P.

- P implies Q.

- P is sufficient for Q.

- Q is necessary for P.

Example: The following sets of statements are equivalent.

- - He will get wet if he stands in the rain.
  - If he stands in the rain, then he will get wet.
- - I will get a ticket for speeding only if I get caught.
  - If I get a ticket for speeding, then I got caught.
- - She takes a bike ride whenever she is upset.
  - If she is upset, then she takes a bike ride.
- - If you get eight hours of sleep, then you will feel good in the morning.

– Feeling good in the morning is necessary for getting eight hours of sleep.

– Getting eight hours of sleep is sufficient for feeling good in the morning.

Note: We will also assume that the statement $\sim Q \Rightarrow P$ (If not Q, then P) is equivalent to each of the following statements:

- P unless Q.

- P without Q.

Example: The following four statements are equivalent.

1. She will not go swimming unless the water is very warm.

2. She will not go swimming without the water being very warm.

3. If the water is not very warm, then she will not go swimming.

4. If she goes swimming, then the water is very warm.

Examples:

Worksheet