Assignment # 4

Due 6-26-06

(1) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or "washer".

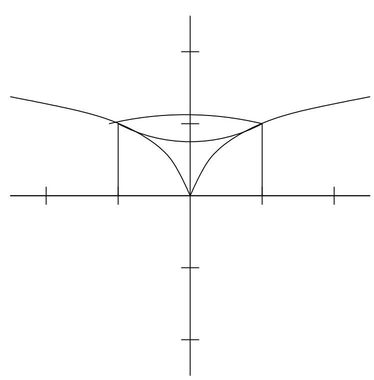
(a) $y = x^{\frac{2}{3}}$, $x = 1$, $y = 0$; about the y-axis.

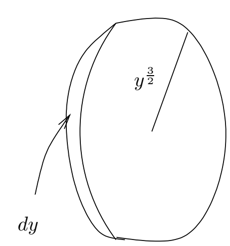
$$y = x^{\frac{2}{3}}$$

$$x = y^{\frac{3}{2}}$$

When $x = 1$

$$y = 1$$





Volume of 1 disk

$$\pi(y^{\frac{3}{2}})^2 dy = \pi y^3 dy$$

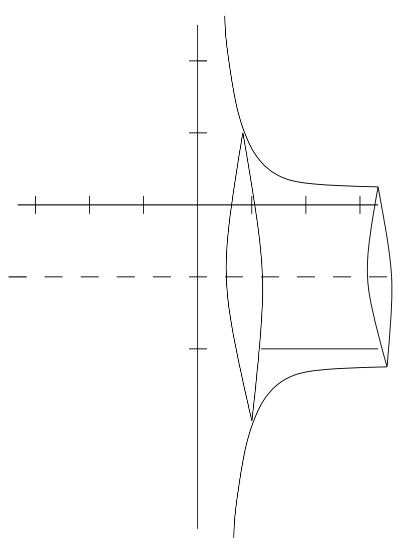
$$\begin{aligned}\text{Inside Volume Total} &= \int_0^1 \pi y^3 dy = \pi \frac{y^4}{4}\Big|_0^1 = \frac{\pi}{4} \\ \text{Wanted Volume Total} &= \pi(1^2)(1) - \frac{\pi}{4} = \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

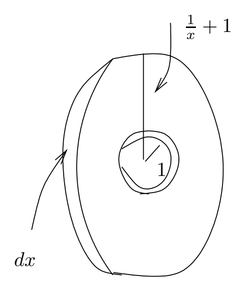
(b) $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 3$; about $y = 1$.

$$y = \frac{1}{x} \qquad y = 0$$

$$x = 1 \qquad x = 3$$

$$\text{about} \quad y = -1$$





Volume of 1 washer

$$\pi((\frac{1}{x} + 1)^2 - 1^2)dx$$

$$
\begin{aligned}
\text{Total Volume} &= \int_1^3 \pi \left[(\frac{1}{x}+1)^2 - 1^2 \right] \; dx \\
&= \pi \int_1^3 \frac{1}{x^2} + \frac{2}{x} + 1 - 1 \; dx
\end{aligned}
$$

$$
= \pi \; -\frac{1}{x} + 2\ln|x| \Big|_1^3
$$

$$
\begin{aligned}
&= \pi \left[-\frac{1}{3} + 2\ln|3| - (-1 + 2\ln|1|) \right] \\
&= \pi \left[\frac{2}{3} + 2\ln(3) \right]
\end{aligned}
$$

(2) Consider the curve

$$y = 2^x$$

(a) Set up the integral that represents the length of the curve on the interval $0 \leq x \leq 3$.

$y = 2^x \;\; 0 \leq x \leq 3$

$$\frac{dy}{dx} = (\ln 2)2^x$$

$$
\begin{aligned}
L &= \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx \\
&= \int_0^3 \sqrt{1 + (\ln 2 \cdot 2^x)^2} \; dx \\
&= \frac{\ln\left(\sqrt{64(\ln(2))^2+1} - 1\right) - \ln\left(\sqrt{(\ln 2)^2+1} - 1\right) + \sqrt{64(\ln 2)^2+1} - \sqrt{(\ln 2)^2+1} - 3\ln(2)}{\ln(2)}
\end{aligned}
$$

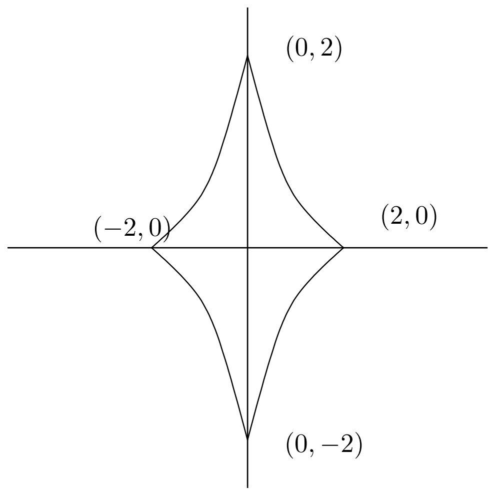
≈ 7.792

(3) The following parametric equation describes an astroid:

$$x = 2\cos^3\theta,$$

$$y = 2\sin^3\theta.$$

(a) Show a sketch of the astroid.



(b) Find the equation of the tangent line to the astroid at the point when $\theta = \dfrac{\pi}{6}$. Show the tangent line on your graph.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{6\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta}$$

$$= -\tan\theta$$

at $\theta = \frac{\pi}{6}$

$$\frac{dy}{dx} = -\tan(\frac{\pi}{6}) = -\frac{\sqrt{3}}{3}$$

$$x = \frac{3\sqrt{3}}{4} \quad \text{and}$$

$$y = \frac{1}{4}$$

So the equation of the tangent line is

$$y - \frac{1}{4} = -\frac{\sqrt{3}}{3}\left(x - \frac{3\sqrt{3}}{4}\right)$$

(c) Find the total length of the astroid. To do this, find the length of the part of the curve (by hand) in quadrant I ($0 \leq \theta \leq \frac{\pi}{2}$) and then multiply this answer by 4.

$$\begin{aligned}
L &= 4\int_0^{\frac{\pi}{2}} \sqrt{(6\sin^2\theta\cos\theta)^2 + (-6\cos^2\theta\sin\theta)^2}\ d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \sqrt{36\sin^4\theta\cos^2\theta + 36\cos^4\theta\sin^2\theta}\ d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \sqrt{36(\sin^2\theta\cos^2\theta)(\sin^2\theta + \cos^2\theta)}\ d\theta \\
&= 4\int_0^{\frac{\pi}{2}} 6\sin\theta\cos\theta\ d\theta
\end{aligned}$$

Let $u = \sin\theta$

$$du = \cos\theta$$

$$\begin{aligned}
&= 24\int_0^1 u\ du \\
&= 24\left(\frac{u^2}{2}\right)\Bigg|_0^1 \\
&= 12
\end{aligned}$$

(4) Consider the function $f(x) = \ln x$, on the interval $[1, 3]$.

(a) Find the average value of f on the given interval.

$$\begin{aligned} a &= \frac{1}{3-1}\int_1^3 \ln x \; dx \\ &= \frac{1}{2}\left(x\ln x - x|_1^3\right) \\ &= \frac{1}{2}\left(3\ln 3 - 3 - ln1 + 1\right) \\ &= \frac{3}{2}\ln(3) - 1 \\ &= \ln\left(3^{\frac{3}{2}}\right) - \ln(e) \\ &= \ln\left(\frac{3\sqrt{3}}{e}\right) \end{aligned}$$

(b) Find c such that $f_{\text{ave}} = f(c)$.

So $c = \frac{3\sqrt{3}}{e}$.

(c) Sketch a graph of f and a rectangle whose area is the same as the area under the graph of f.