Solutions to Assignment # 3

(1) Determine whether the integral converges or diverges. Find the value of the integral if it converges.

(a) $\int_0^1 x^{-\frac{4}{5}}\, dx$

Converges

$$\begin{aligned}\int_0^1 x^{-\frac{4}{5}}\, dx &= \lim_{c\to 0^+}\int_c^1 x^{-\frac{4}{5}}\, dx\\ &= \lim_{c\to 0^+} 5x^{\frac{1}{5}}\Big|_c^1\\ &= \lim_{c\to 0^+} 5(1-c^{\frac{1}{5}})\\ &= 5\end{aligned}$$

(b) $\int_0^1 x^{-\frac{6}{5}}\, dx$

Diverges

$$\begin{aligned}\int_0^1 x^{-\frac{6}{5}}\, dx &= \lim_{c\to 0^+}\int_c^1 x^{-\frac{6}{5}}\, dx\\ &= \lim_{c\to 0^+} -5x^{-\frac{1}{5}}\Big|_c^1\\ &= \lim_{c\to 0^+} -5+5c^{-\frac{1}{5}} = \infty\end{aligned}$$

(c) $\int_1^\infty x^{-\frac{4}{5}}\ dx$

Diverges.

$$\begin{aligned}\int_1^\infty x^{-\frac{4}{5}}\ dx &= \lim_{b\to infty}\int_1^b x^{-\frac{4}{5}}\ dx\\ &= \lim_{b\to\infty} 5x^{\frac{1}{5}}\Big|_1^b\\ &= \lim_{b\to\infty} 5(b^{\frac{1}{5}}-1)\\ &= \infty\end{aligned}$$

(d) $\int_1^\infty x^{-\frac{6}{5}}\ dx$

Converges.

$$\begin{aligned}\int_1^\infty x^{-\frac{6}{5}}\ dx &= \lim_{b\to infty}\int_1^b x^{-\frac{6}{5}}\ dx\\ &= \lim_{b\to\infty} -5x^{-\frac{1}{5}}\Big|_1^b\\ &= \lim_{b\to\infty} -5b^{-\frac{1}{5}}\\ &= 5\end{aligned}$$

(e) $\int_{-4}^4 \frac{2x}{x^2-1}\ dx$

Diverges. Notice that

$$\int_{-4}^4 \frac{2x}{x^2-1}\ dx = \int_{-4}^{-1} \frac{2x}{x^2-1}\ dx + \int_{-1}^{0} \frac{2x}{x^2-1}\ dx + \int_{0}^{1} \frac{2x}{x^2-1}\ dx + \int_{1}^{4} \frac{2x}{x^2-1}\ dx$$

but

$$\begin{aligned}\int_1^4 \frac{2x}{x^2-1}\ dx &= \lim_{c\to1^+}\int_c^4 \frac{2x}{x^2-1}\ dx\\ &= \lim_{c\to1^+} \ln(x^2-1)\Big|_c^4\\ &= \lim_{c\to1^+} \ln(15)-\ln(c^2-1)=\infty\end{aligned}$$

So the integral diverges.

(f) $\int_{-\infty}^{\infty} \frac{1}{x^2-1}\, dx$

Diverges.

$$\int_{-\infty}^{\infty} \frac{1}{x^2-1}\, dx = \int_{-\infty}^{0} \frac{1}{x^2-1}\, dx + \int_{0}^{1} \frac{1}{x^2-1}\, dx + \int_{1}^{2} \frac{1}{x^2-1}\, dx + \int_{2}^{\infty} \frac{1}{x^2-1}\, dx$$

But,

$$\begin{aligned}\int_{1}^{2} \frac{1}{x^2-1}\, dx &= \lim_{c\to 1^+} \int_{c}^{2} \frac{1}{x^2-1}\, dx \\ &= \lim_{c\to 1^+} \frac{1}{2}\ln\left(\frac{x-1}{x_+1}\right)\Bigg|_c^2 \qquad \text{(Using Partial fractions)} \\ &= \lim_{c\to 1^+} \frac{1}{2}\ln\left(\frac{1}{3}\right) - \frac{1}{2}\ln\left(\frac{c-1}{c+1}\right) = \infty\end{aligned}$$

(g) $\int_{0}^{1} \frac{2}{x\sqrt{1-x^2}}\, dx$

Diverges.

$$\int_{0}^{1} \frac{2}{x\sqrt{1-x^2}}\, dx = \int_{0}^{\frac{1}{2}} \frac{2}{x\sqrt{1-x^2}}\, dx + \int_{\frac{1}{2}}^{1} \frac{2}{x\sqrt{1-x^2}}\, dx$$

But

$$\begin{aligned}\int_{0}^{\frac{1}{2}} \frac{2}{x\sqrt{1-x^2}}\, dx &= \lim_{c\to 0^+} \int_{c}^{\frac{1}{2}} \frac{2}{x\sqrt{1-x^2}}\, dx \\ &= \lim_{c\to 0^+} -2\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\Bigg|_c^{\frac{1}{2}} \qquad \text{(By Integral 34 in the back of the book)} \\ &= \infty\end{aligned}$$

(2) Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

$$x + y^2 = 2, \quad x + y = 0$$

$$x + y^2 = 2$$

$$y = -x$$

$$x = 2 - y^2$$

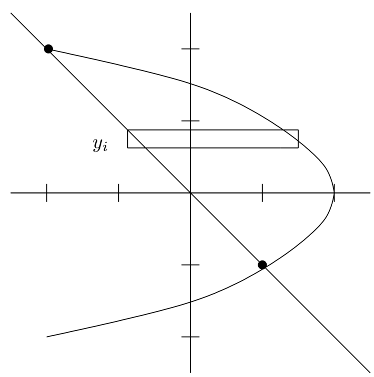
$$x = -y$$

$$2 - y^2 = -y$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y = 2 \quad \text{and} \quad y = -1$$



$$\text{Area} \quad = \int_{-1}^{2} (2 - y^2) - (-y)\ dy = \frac{9}{2}$$

(3) Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

$$y = e^x, \quad y = 2 - x^2$$

$$y = e^x$$

$$y = 2 - x^2$$

Points of intersection

$$x \approx -1.315974$$

$$x \approx .53727445$$

$$\text{Area} \quad \approx \int_{-1.315974}^{.53727445} (2 - x^2) - e^x \; dx = 1.452013983$$

(4) Consider the curve $y = \dfrac{1}{x^2}$, $1 \leq x \leq 4$.

(a) Find the number a such that the line $x = a$ bisects the area under the curve.

$$\int_1^a \frac{1}{x^2} \; dx = \int_a^4 \frac{1}{x^2} \; dx$$

$$a = \frac{8}{5}$$

(b) Find the number b such that the line $y = b$ bisects the area in part (a).

$$\int_0^{\frac{1}{16}} 4 - 1 \; dy + \int_{\frac{1}{16}}^{b} \frac{1}{\sqrt{y}} - 1 \; dy = \int_b^1 \frac{1}{\sqrt{y}} - 1 \; dy$$

$$b = \frac{(\sqrt{6} - 4)^2}{16}$$