Math 2411 Calculus II
Practice Final, Summer 2006

Name: **Solutions**

1. The first five terms of a sequence are written below.

$$\frac{1}{3}, \frac{4}{4}, \frac{9}{5}, \frac{16}{6}, \frac{25}{7}, \frac{36}{8}, \ldots$$

   (a) Write an expression which describes the rule for the nth term **(4 pts)**. $a_n = \left\{ \frac{n^2}{n+2} \right\}_{n=1}^{\infty}$

   (b) Determine if the sequence converges or diverges. If the sequence converges, find the value it converges to. Show your work. **(4 pts)**

$$\lim_{n\to\infty} \frac{n^2}{n+2} = \infty$$

Therefore the sequence diverges.

2. Consider the series defined as $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$

   (a) Do a **partial fraction decomposition** to rewrite the rule for a_n of the series. **(4 pts)**

$$\begin{aligned} \frac{2}{(2n-1)(2n+1)} &= \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{2An + A + 2Bn - B}{4n^2-1} \\ 2A + 2B &= 0 \\ A - B &= 2 \\ A &= 1 \\ B &= -1 \\ \frac{2}{4n^2-1} &= \frac{1}{2n-1} - \frac{1}{2n+1} \end{aligned}$$

   (b) Write the first **4 partial sums** of the series, then determine a rule for the n**th partial sum. (4 pts)**

$S_1 = \frac{1}{1} - \frac{1}{3}$

$S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = 1 - \frac{1}{5}$

$S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) = 1 - \frac{1}{7}$

$S_4 = \left(1 - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$S_n = 1 - \frac{1}{2n+1}$

   (c) Find the sum S of the original series, if it exists. **(4 pts)**

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2n+1} = 1$$

So the Series converges to 1.

3. Find the sum, if it exists, of the geometric series $\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{4^n}$ **(6 pts)**

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{4^n} &= \sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{n-1} \\
a &= \frac{3}{2} \\
r &= \frac{3}{4} < 1
\end{aligned}$$

So the series converges. Using the sum formula for geometric series we get

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3^n}{4^n} = \frac{\left(\frac{3}{2}\right)}{\left(1 - \frac{3}{4}\right)} = \frac{3}{2} \cdot \frac{4}{1} = 6$$

Series Tests:

1. nth Term Test
2. Special Series Tests:
   - a. Geometric Series
   - b. p-series
3. Ratio Test
4. Limit Comparison Test
5. Integral Test
6. Alternating Series Test

4. Determine if the following series **converge** or **diverge**. State the test you are using and explain your work. **(6 pts each)**

(a) $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$\lim_{n \to \infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$ So $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ diverges by the nth term test.

(b) $\sum_{n=1}^{\infty} \frac{1}{[(n+1)\ln(n+1)][\ln(\ln(n+1))]}$

Consider $f(x) = \frac{1}{(x+1)(\ln(x+1))(\ln(\ln(x+1)))}$.

$$\begin{aligned}
\text{Let } u &= \ln(\ln(x+1)) \\
du &= \frac{1}{\ln(x+1)} \cdot \frac{1}{x+1} \\
\text{So } \int_1^{\infty} f(x)\,dx &= \int_{\ln(\ln(2))}^{\infty} \frac{1}{u}\,du \\
&= \ln(u)\Big|_{\ln(\ln(2))}^{\infty} = \infty
\end{aligned}$$

So the series diverges by the integral test.

(c) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

Since $\frac{3}{2} > 1$ the series converges by the p-series test.

5. Determine whether the **alternating series** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$ is **absolutely convergent, conditionally convergent, or divergent.** Show all work. **(8 pts)**

Let $f(x) = \dfrac{\ln(x+1)}{x+1}$

$$f'(x) = \frac{\frac{1}{x+1}(x+1) - \ln(x+1)}{(x+1)^2} = \frac{1-\ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq e-1.$$

So the terms are eventually monotoncally decreasing. Also notice that

$$\lim_{n\to\infty}\frac{\ln(n+1)}{n+1} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{1} = 0.$$

So the series converges by the alternating series test.

Now $\dfrac{\ln(n+1)}{n+1} \geq \dfrac{1}{n+1}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n+1}$ is divergent, so $\sum_{n=1}^{\infty}\dfrac{\ln(n+1)}{n+1}$ is divergent. So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$ is conditionally convergent.

6. Find the **interval of convergence** for the power series $\sum_{n=1}^{\infty}\dfrac{(x-2)^{n+1}}{(n+1)\cdot 4^{n+1}}$. (Don't forget to check endpoints.) **(8 pts)**

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x-2)^{n+2}}{(n+2)4^{n+2}}\cdot\frac{(n+1)4^{n+1}}{(x-2)^{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{2(n+2)}\right| \\ &= \left|\frac{x-2}{4}\right| < 1 \Rightarrow \\ -4 &< x-2 < 4 \Rightarrow \\ -2 &< x < 6\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$$

This series converges by the alternating series test.

$$\sum_{n=1}^{\infty}\frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{n+1}$$

This series diverges by the p-series test. So the interval of convergence is

$$-2 \leq x < 6$$

7. Use the **definition of the Taylor Series** and write the first 3 non-zero terms for $f(x) = e^{(-2x)}$ centered at $c = 0$. Then write the series using sigma notation. **(8 pts)**

$$\begin{aligned}
f(x) &= e^{-2x} \quad f(0) = 1 \\
f'(x) &= -2e^{-2x} \quad f'(0) = -2 \\
f''(x) &= 4e^{-2x} \quad f''(0) = 4 \\
f'''(x) &= -8e^{-2x} \quad f'''(0) = -8 \\
&\vdots \\
f^{(n)}(x) &= (-1)^n 2^n e^{-2x} \quad f^{(n)}(0) = (-1)^n 2^n \\
f(x) &= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}
\end{aligned}$$

8. Use the Taylor Series $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, and write the first 4 terms and the series in sigma notation for: **(6 pts)**

$$\begin{aligned}
\cosh x = \frac{1}{2}(e^x + e^{-x}) &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) \\
&= \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!}\frac{x^6}{6!} + \cdots \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

9. Use the Taylor Series $\cos x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, and write the first 4 terms and the series in sigma notation for: **(6 pts)**

$\int \cos(\sqrt{x})\, dx$

$$\begin{aligned}
\cos(\sqrt{x}) &= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \\
\text{So } \int \cos(\sqrt{x})\, dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)(2n)!} \\
\int \cos(\sqrt{x})\, dx &= C + x - \frac{x^2}{(2)(2!)} + \frac{x^3}{(3)(4!)} - \frac{x^4}{(4)(6!)} \cdots
\end{aligned}$$