Calculus I **Solutions**

Exam #2 - Fall 2004

(1) Differentiate the following functions: (**6 pts each**)

(a)

$$k(x) = \mathrm{e}^{2x} \cdot \sin(x) \cdot 4x^3$$

Solution:

$$k'(x) = 2e^{2x} \cdot \sin(x) \cdot 4x^3 + e^{2x} \cdot \cos(x) \cdot 4x^3 + e^{2x} \cdot \sin(x) \cdot 12x^2$$

(b)

$$f(x) = \sin(x^2 - 1) + \cos(\ln(x))$$

Solution:

$$\begin{aligned} f'(x) &= \cos(x^2-1) \cdot 2x + \frac{-\sin(\ln(x))}{x} \\ &= 2x\cos(x^2-1) - \frac{\sin(\ln(x))}{x} \end{aligned}$$

(c)

$$f(x) = \frac{2\sin(3x)}{\ln(x+2)}$$

Solution:

$$\begin{aligned} f'(x) &= \frac{(6\cos(3x))(\ln(x+2)) - \frac{2\sin(3x)}{x+2}}{(\ln(x+2))^2} \\ &= \frac{(x+2)(6\cos(3x))(\ln(x+2)) - 2\sin(3x)}{(x+2)(\ln(x+2))^2} \end{aligned}$$

(d)

$$h(x) = \mathrm{e}^{x^2} + 14^x - \mathrm{e}^{\pi}$$

Solution:

$$h'(x) = 2xe^{x^2} + \ln(14) \cdot 14^x$$

(2) Differentiate the following functions: (**8 pts each**)

(a)

$$g(x) = \tan(2x^2 - 1) + \sin^2(x) + \cos^2(x)$$

Solution:

$$g(x) = \tan(2x^2 - 1) + 1$$

$$g'(x) = 4x\sec^2(2x^2 - 1)$$

(b)

$$g(x) = \cos(\mathrm{e}^x)(\arccos(x))$$

Solution:

$$g'(x) = -e^x \sin(e^x)(\arccos(x)) - \frac{\cos(e^x)}{\sqrt{1-x^2}}$$

(3) Given the following expression, find $\frac{dy}{dx}$. (**8pts**)

$$x^4 + 3x^2y + y^2 = \frac{222}{13}$$

Solution:

$$\begin{aligned}
4x^3 + 6xy + 3x^2\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\
3x^2\frac{dy}{dx} + 2y\frac{dy}{dx} &= -4x^3 - 6xy \\
\frac{dy}{dx}(3x^2 + 2y) &= -4x^3 - 6xy \\
\frac{dy}{dx} &= \frac{-4x^3 - 6xy}{3x^2 + 2y}
\end{aligned}$$

(4) Find the equation of the tangent line to the curve $y = x^{3x^2}$ at the point $(1,1)$. (**10 pts**) **Solution:**

$$\begin{aligned}
\ln(y) &= \ln(x^{3x^2}) \\
\ln(y) &= 3x^2\ln(x) \\
\frac{1}{y}\frac{dy}{dx} &= 6x\ln(x) + \frac{3x^2}{x} \\
\frac{1}{y}\frac{dy}{dx} &= 6x\ln(x) + 3x \\
\frac{dy}{dx} &= y(6x\ln(x) + 3x) \\
\frac{dy}{dx}|_{(1,1)} &= 1(6(1)\ln(1) + 3(1)) \\
&= 1(6(0) + 3) = 3
\end{aligned}$$

So the equation of the tangent line at $(1,1)$ is

$$y - 1 = 3(x - 1)$$

(5) On what intervals is the function $f(x) = \mathrm{e}^{-x^2}$ concave up? (**10 pts**) **Solution:**

$$f'(x) = -2x\mathrm{e}^{-x^2}$$

$$\begin{aligned} f''(x) &= -2\mathrm{e}^{-x^2} + (-2x)(-2x)\mathrm{e}^{-x^2} \\ &= -2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2} \end{aligned}$$

$$\begin{aligned} f \text{ concave up } &\Rightarrow f''(x) > 0 \\ &\Rightarrow -2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2} > 0 \\ &\Rightarrow (4x^2 - 2)\mathrm{e}^{-x^2} > 0 \\ &\Rightarrow 4x^2 - 2 > 0 \\ &\Rightarrow 4x^2 > 2 \\ &\Rightarrow x^2 > \frac{1}{2} \\ &\Rightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \end{aligned}$$

(6) The equation of a particular ellipse is given by $3(x-4)^2 + 2(y-2)^2 = 14$ Find the equation of the tangent line to this ellipse at the point $(6, 3)$. (**10 pts**) **Solution**

$$\begin{aligned} 6(x-4) + 4(y-2) \cdot \frac{dy}{dx} &= 0 \\ 4(y-2) \cdot \frac{dy}{dx} &= -6(x-4) \\ \frac{dy}{dx} &= -\frac{6(x-4)}{4(y-2)} \\ \frac{dy}{dx} &= -\frac{3(x-4)}{2(y-2)} \\ \frac{dy}{dx}|_{(6,3)} &= -\frac{3(2)}{2(1)} \\ &= -3 \end{aligned}$$

So the equation of our tangent line is

$$y - 3 = -3(x - 6)$$

(7) Use the quotient rule to derive the derivative of $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Simplify your answer!

(**Note: A correct answer with no work will receive no credit!**)

Solution:

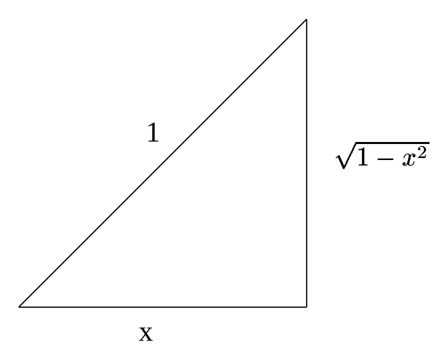
$$\frac{d}{dx}(\cot(x)) = \frac{d}{dx}(\frac{\cos(x)}{\sin(x)}) = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$$
$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)}$$
$$= \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

(8) Let $y = \arccos(x)$. Derive a formula in terms of x for $\frac{dy}{dx}$.

(**Note: A correct answer with no work will receive no credit!**)

Solution:

$$y = \arccos(x) \Rightarrow$$
$$x = \cos(y)$$
$$1 = -\sin(y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$



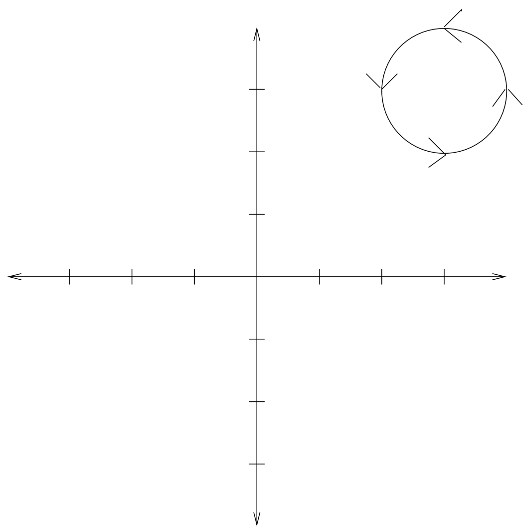
$$\text{So, } \sin(y) = \sqrt{1-x^2} \Rightarrow$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(9) **BONUS 10 pts**

Given The parametric equation

$$x = 2 + \cos t \qquad y = 3 + \sin t \qquad 0 \le t \le 2\pi$$

(a) Sketch the grah of this curve and use arrows to indicate the direction in which the curve is traced as t increases.



(b) Find $\frac{dy}{dx}$ 3 different ways **Solutions:**

(i)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

(ii) Also,

$$\cos t = x - 2$$

$$\sin t = y - 3$$

$$\sin^2 t + \cos^2 t = 1$$

$$(x-2)^2 + (y-3)^2 = 1$$

$$\Rightarrow$$

$$2(x-2) + 2(y-3)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(x-2)}{(y-3)}$$

(iii)

$$\begin{aligned}(y-2)^2 &= 1-(x-2)^2\\ y-3 &= \pm\sqrt{1-(x-2)^2}\\ y &= 3 \pm \sqrt{1-(x-2)^2}\\ \frac{dy}{dx} &= \pm\frac{1}{2\sqrt{1-(x-2)^2}} \cdot -2(x-2)\\ &= \mp\frac{x-2}{\sqrt{1-(x-2)^2}}\end{aligned}$$

(iv)

$$\begin{aligned}t &= \arccos(x-2)\\ y &= 3+\sin(\arccos(x-2))\\ \frac{dy}{dx} &= \cos(\arccos(x-2)) \cdot \frac{-1}{\sqrt{1-(x-2)^2}}\\ &= \frac{-(x-2)}{\sqrt{1-(x-2)^2}}\end{aligned}$$

(c) For what t values does the curve have a horizontal tangent line? **Solution:**

$$\begin{aligned}\frac{dy}{dt} = 0 &\text{ and } \frac{dx}{dt} \neq 0\\ \cos t = 0 &\text{ and } -\sin t \neq 0\\ t &= \frac{\pi}{2}, \frac{3\pi}{2}\end{aligned}$$

(d) For what t values does the curve have a vertical tangent line? **Solution:**

$$\begin{aligned}\frac{dx}{dt} = 0 &\text{ and } \frac{dy}{dt} \neq 0\\ -\sin t = 0 &\text{ and } \cos t \neq 0\\ t &= 0, \pi\end{aligned}$$