Stochastic Processes (Math 5780)

FINAL EXAM. December 3, 2008

Instructions. This is a take-home exam. You have a week to complete it. The exam due date is December 10. Please, write legibly and provide explanation for your conclusions.

1. (20pt) ex. 5.6 or ex. 7.6.

2. (20pt) ex. 4.5.

3. (20pt) Let (S_n) be a simple random walk on $\mathbb{Z}$. For a fixed $a \in \mathbb{N}$ define $T = \min\{n \in \mathbb{N} : |S_n| = a\}$.

   (a) Show that $M_n = S_n^2 - n$ and $N_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n$ are martingales.

   (b) Use the optional sampling theorem in order to find $\mathbf{E}(T)$ and $\mathbf{E}(T^2)$. (You don't need to prove that the theorem applies.)

4. (20pt) Consider a discrete-time Markov chain on state space $\{1, 2, \dots, N\}$ with transition probabilities p_{ij} which is reversible with respect to stationary probabilities π_i . The chain truncated to the state space $\{1, 2, \dots, M\}$ where $M < N$ is defined by eliminating transitions to states i with $i > M$, i.e., whenever the original chain attempts to move to one of these states the truncated chain "stays put". In other words, the transition probabilities for the truncated chain are $\overline{p}_{ii} = p_{ii} + \sum_{k=M+1}^{N} p_{ik}$ and $\overline{p}_{ij} = p_{ij}$ for $i, j \in \{1, \dots, M\}$. Show that the truncated chain is also time reversible with respect to the distribution $\overline{\pi}_i = \pi_i / \sum_{k=1}^{M} \pi_k$.

5. (20pt) Suppose $X_t = (X_{1,t}, X_{2,t})$ is a two-dimensional standard Brownian motion. Let $Y_{1,t}$ denote the orthogonal projection of X_t onto the bisector $y = x$ and let $Y_{2,t}$ denote the orthogonal projection of X_t onto the bisector $y = -x$. Show that $Y_{1,t}$ and $Y_{2,t}$ are independent Brownian motions and find their variances.

- **Bonus (10pt)**.
  Check the hypotheses of the optional sampling theorem in problem 3. (Hint: ex. 1.7)