NAME:

Probability (Math 4810/5310)

SAMPLE FINAL EXAM.

Instructions. This is a 2 hour exam. You may use the textbook and your notes. No use of calculators or other electronic devices is allowed.

In order to qualify for a grade of C or better you must give correct answers to at least 7 out of 10 C-level questions. In order to qualify for a grade of B or higher you must give correct answers to at least 7 out of 10 C-level questions and to score at least 70% on the B-level questions. In order to get an A you must give correct answers to at least 7 out of 10 C-level questions, to score at least 70% on the B-level questions, and to score at least 90% on the A-level questions. For full credit on the B-level and A-level questions you must highlight your answers, show work to demonstrate your methods and provide justification.

- **C-level questions.**

  Please, circle the correct answer.

  1. Consider the experiment consisting of first rolling a die and flipping a coin after that. What is the number of outcomes of this experiment ?

     A. 6
     B. 7
     C. 12
     D. None of the above.

     Answer. C.

  2. Suppose A and B are independent events with respective probabilities 1/2 and 1/3. What is the conditional probability $P(A|B)$?

     A. 1
     B. 0
     C. 1/2
     D. None of the above.

     Answer. C.

  3. Suppose X and Y are independent Bernoulli random variables with respective parameters p and q. What is the variance of $X + Y$?

     A. $p(1 - p) + q(1 - q)$
     B. $p + q$
     C. $p^2 + q^2$
     D. None of the above

     Answer. A.

  4. Suppose X is a continuous random variable with density $f(x) = ax^2$ for $x \in [0, 1]$ and $f(x) = 0$ otherwise. Find the value of a.

     A. 3

B. 2

C. -3

D. None of the above

Answer. A.

5. Suppose Z is a Poisson random variable with parameter $\lambda = 2$. What is the variance of $2Z$?

A. 8

B. 4

C. 10

D. None of the above

Answer. A.

6. Let X and Y be independent random variables uniformly distributed on $[0, 1]$. Find $P(X^2 + Y^2 \leq 1)$.

A. π

B. $\pi/4$

C. $\pi/2$

D. None of the above

Answer. B.

7. Suppose X is a continuous random variable with density $f(x) = e^x/2$ for $x < 0$ and $f(x) = e^{-x}/2$ for $x > 0$. Find $P(|X| > 1)$.

A. e^{-1}

B. 0

C. $1 - e^{-1}$

D. None of the above

Answer. A.

8. Let X represent a geometric random variable with parameter $1/2$. Evaluate $P(X \leq 2)$.

A. 1/2

B. 1

C. 3/4

D. None of the above

Answer. C.

9. Let Z represent a standard normal random variable. Evaluate $P(Z \leq -2)$ approximately.

A. .03

B. .5

C. .3

D. None of the above

Answer. A.

10. Let X represent the number of heads in 4 flips of a fair coin. Evaluate the second moment of X.

A. 1

B. 6

C. 4

D. None of the above

Answer. D. $E(X^2) = (E(X))^2 + V(X) = (4 \cdot 1/2)^2 + 4 \cdot (1/2 \cdot 1/2) = 5$.

- **B-level questions**

1. (10pt) The probability that a randomly chosen male has a circulation problem is 0.25 . Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

   Answer. Bayes formula:

$$P(C|S) = \frac{P(S|C)P(C)}{P(S|C)P(C) + P(S|\tilde{C})P(\tilde{C})} = \frac{P(S|C) \cdot .25}{P(S|C) \cdot .25 + (P(S|C)/2) \cdot .75} = .4\,.$$

2. (10pt) At University of Colorado Denver 20% of students major in economics, 5% of students major in Economics and English, and 70% of students major in neither Economics nor English. Find the proportion of students who major in English. Answer.

$$P(Ec \cup En) = P(Ec) + P(En) - P(Ec \cap En).$$

$$P(Ec \cup En) = 1 - .7 = .3.$$

$$P(En) = .3 - .2 + .05 = .15.$$

3. (10pt) Daily demand for a product is a random variable with a mean of 20 and a standard deviation of 5. Suppose that the demands on different days are independent and identically distributed. How many items should there be in stock at the beginning of the month so that there is a 0.9 probability that the demand for the month will not exceed this number? Assume a month has 30 days.

   Answer. Demand D is approximately normal with mean 600 and variance 150:

$$P(D \geq x) = P(D^* \geq (x - 600)/5\sqrt{30}) \approx P(Z \geq (x - 600)/27.39) \leq .1.$$

   Hence, $(x - 600)/27.39 \geq 1.29$, $x \geq 636$.

4. (10pt) Suppose that the time until the next telemarketer calls you is distributed as an exponential random variable. If the chance of your getting such a call from 6 p.m. to 7 p.m. is .5, what is the chance that you'll get such a call during the next two hours ?

   Answer. Let X be exponential with λ. Then $P(X \leq 1) = 1 - e^{-\lambda} = .5$. By the memoryless property, we need $P(X \leq 2) = 1 - e^{-2\lambda}$. It is $1 - .5^2 = .75$.

5. (10pt) Suppose X and Y are continuous random variables with joint pdf $f(x, y) = x + y$ for $x \in [0, 1]$ and $y \in [0, 1]$, and $f(x, y) = 0$ elsewhere. Justify that it is a valid pdf, determine the marginal pdf of X, and evaluate the third moment of X.

Answer. Check that $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$.

$$f_X(x) = \int_0^1 f(x,y)\,dy = x + 1/2.$$

$$E(X^3) = \int_0^1 x^3(x+1/2)\,dx = 1/5 + 1/8 = 13/40.$$

6. (10pt) Suppose X is uniform on the interval $[1,2]$. Compute the cdf, the pdf, and expected value of the random variable $Y = 1/X$.
   Answer. The range of Y is $[1/2, 1]$. The cdf is

   $$F(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 2 - 1/y.$$

   The pdf is

   $$f(y) = F'(y) = 1/y^2\,.$$

   $$E(Y) = \int_{1/2}^1 y \cdot 1/y^2\,dy = \ln 2\,.$$

7. (10pt) Suppose there are 100 families in Denver with 5 children. How many of those families would you expect to have all 5 daughters? Assume that the probability of a child being a girl is .5.
   Answer. Probability of 5 daughters is $.5^5$. $E(X) = 100 \cdot .5^5 \approx 3.1$.

- **A-level questions**

1. (10pt) A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails. Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

   $$f(x,y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } 0 < x < y < \infty, \\ 0, & \text{otherwise} \end{cases}$$

   What is the expected time at which the device fails?
   Answer.

   $$E(X+Y) = \int_0^\infty \int_x^\infty (x+y)6e^{-x}e^{-2y}\,dy\,dx = 2\,.$$

2. Ten different books are to be arranged on a shelf at random (therefore, all possible permutations of the books can be taken to be equally likely). Three of the books are written by Dostoyevsky and the other seven by other authors. Find the probability that all three Dostoyevsky books will be placed one next to the other.

   Answer. There are 10! permutations in total. There are $7! \cdot 3!$ permutations of the books when the D. books occupy three fixed consecutive positions. The are 8 ways to pick three consecutive positions. The number of arrangements when the D books are next to each other is $7! \cdot 3! \cdot 8$. The probability in question is

   $$\frac{7! \cdot 3! \cdot 8}{10!} = 1/15\,.$$